8.1 Tonal systems

The strings of the guitar are tuned to E-A-D-G-B-E (for standard tuning). These notes are (inter alia) elements of a musical scale that itself is an element of a tonal system (other expressions used are pitch space, or system of tonality). The latter is understood as the (theoretically unlimited) set of all ordered notes, and includes defining the individual distances between pitches. In Western music, the tonal system with 12 steps is predominant, with the musical scale formed of 12 notes. Deviating from this system are, for example, the pentatonic system (based on merely 5 notes), and the diatonic system. The distance between the notes (the frequency relationships) can be derived from the rules of the tonal system, and from this we obtain the design rules of the guitar, and the tuning rules of the individual strings. In this context, basic knowledge of vibration engineering proves to be helpful.

It is mainly **transversal waves** that propagate on the guitar string; they are reflected at the termination of the free string (nut, bridge). A single-frequency excitation of the string leads to particularly strong vibration patterns at specific frequencies (Eigen-modes at the Eigen-frequencies i.e. natural modes at the natural frequencies). The lowest frequency at which such an **Eigen-mode** occurs is the fundamental frequency of the string. In a simplified view, all higher Eigen-frequencies are integer multiples of this fundamental frequency; a more detailed analysis shows a slight spreading of the frequencies (see Chapter 1).

Fig. 8.1 shows the first three Eigen-modes of an ideal string vibrating in a single-frequency fashion. If the excitation of the string is not with a single frequency but with a plurality of frequencies (e.g. via an impulse), the superposition of many of these Eigen-modes may lead to the formation of a complex vibration-pattern. Each one of the **Eigen-modes** (in theory there is a an infinite number of them) is characterized by four individual parameters: its **Eigen-frequency** that for the n-th Eigen-mode corresponds (in the dispersion-free string) to the n-fold fundamental frequency (n being an integer number); its **amplitude** and **phase**, and its **direction of vibration**. Of these 4 mode-specific quantities, only the frequency shall be considered in the following. Arbitrarily choosing 100 Hz as the fundamental frequency, the frequencies of the higher-order partials ($n > 1$) are 200 Hz, 300 Hz, 400 Hz, etc. Halving the length of the string while maintaining an equal tension-force yields twice the fundamental frequency, with the frequencies of the partials now 200 Hz, 400 Hz, 600 Hz, 800 Hz, etc.

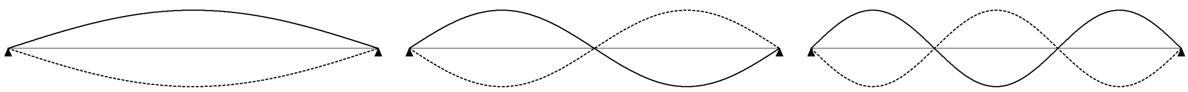

Fig. 8.1: The first three Eigen-modes of an ideal string.
Left: fundamental (1st partial), center: first overtone (2nd harmonic); right: 2nd overtone (3rd partial).

The individual partials do generate individual auditory perceptions in the sense that a multitude of tones becomes audible as a *single* string is plucked. Rather, the pitches of the partials (perceived on a largely subconscious processing plane) blend to form a single **pitch of the string**, with only this pitch being perceived consciously – given favorable conditions. The pitch of the plucked string corresponds *approximately* to the pitch generated by the fundamental vibration, but it is not identical♣. There are small deviations between the two – but for our first basic considerations the deviations shall not be regarded.

♣ The higher-order partials (overtones, $n > 1$) change the pitch of the string only to a minor degree, but they do contribute substantially to the timbre – which is not considered here.

The string mentioned in the example having a fundamental frequency of 100 Hz, and the string shortened by half (fundamental frequency 200 Hz) each generate a tone designated T100 and T200, respectively. Played one after the other in direct comparison, T100 and T200 sound very similar – this is not actually surprising since the frequencies of the partials contained in T200 represent a subset of those contained in T100. This example may be extended by subjecting the halved string (T200) to another halving (T400). The resulting frequencies of the partials (400 Hz, 800 Hz, 1200 Hz, etc.) are again a subset of the frequencies of the partials contained in T100 and T200. Further halving of the string length gives corresponding results. All notes generated by such halving (or doubling) sound very similar, although their pitches differ markedly. Since the frequency relation generated by halving and doubling of the string lengths (2:1 and 1:2, respectively) are designated **octaves** in the musical context, the resulting notes are called **octave-related**. The high degree of auditory relationship between two notes distanced by an octave has led to designating such notes with the same letter. For example, the **reference note** used for tuning to standard ("concert") pitch is internationally as a rule designated A_4, with the note one octave above being designated A_5. However, depending on the national context there are also variations to this system of designations, e.g. a^1 (or a'), and a^2 (or a''), respectively.

8.1.1 The Pythagorean tonal system

Continued halving of the string-length is a first step towards generating related notes of differing fundamental frequency. Following this approach, we find notes with corresponding frequencies of partials also when **reducing the string-length to one third**. The partials of the resulting note (designated T300) are located at 300 Hz, 600 Hz, 900 Hz, 1200Hz, etc. However, compared to T200 now only the frequencies of every other (even-numbered) partial is in correspondence, namely 600 Hz, 1200 Hz, etc. (**Fig. 8.2**). The fundamental frequency of the string reduced to $1/3^{rd}$ in length relates to the fundamental frequency of the halved string, as would 3:2; this frequency relation (frequency interval) is called, in musical terms, a **fifth**. For the associated notes, the concept of **fifth-relationship** is derived from this. Compared to the octave-relationship, the fifth-relationship is less pronounced.

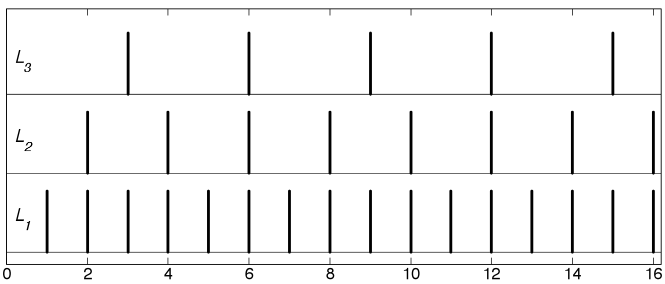

Fig. 8.2: Spectra of partials of strings with the relative lengths: L_1 = 1, L_2 = 1/2, L_3 = 1/3. Abscissa: normalized frequency; ordinate: amplitudes (arbitrary)

Applying jumps of fifths and octaves in combination allows for the generation of a multitude of notes that all are more or less related. Already in the ancient world a tonal system (among many others) was constructed from octave- and fifth- intervals; after its protagonist Pythagoras (ca. 530 B.C.), it is named the **Pythagorean tonal system**. In theory, an infinite number of different notes could be generated with it. However, in practice we arrive at a prominent end point after 12 jumps of one fifth each: after 12 subsequent intervals of one fifth each, the resulting frequency relationship is $1{,}5^{12} = 129{,}74_6$. This brings it close to the 7^{th} octave, the frequency relationship of which amounts to $2^7 = 128$.

The small difference between these two values of 129,74₆ / 128 = 1,013₆ is called the **Pythagorean comma** in music theory. From the sequencing of fifths, and from octave shifts, all notes of Western music can be generated. In this approach, the frequencies of the notes positioned at a distance of a fifth are shifted by a number of octaves until all frequencies are located within one base octave. Starting from the arbitrarily chosen initial frequency 100 %, the following rounded (!) frequencies result (in order to be able to more easily interpret the frequencies, they are given in % to begin with; the corresponding frequencies are listed in Chapter 8.1.3):

100 –	150 –	225 –	338 –	506 –	759 –	1139 –	1709 –	2563 –	3844 –	5767 –	8650 –	12975	%.
100	150	113	169	127	190	142	107	160	120	180	135	203	%.
C	G	D	A	E	H	F#	C#	G#	D#	A#	E#	B#	

The first line in this table holds the ascending frequencies of fifths, the second line includes the corresponding frequencies in the base octave. The designation of the notes is given in the third line (# stands for 'sharp'). For example, 2563 % needs to be shifted (towards lower frequencies) by four octaves in order to arrive at 160%: $2563 / 2^4 = 160$. Rearranging the frequencies in the second line in monotonously ascending order, the sequence of frequencies of a scale results (values rounded off):

100 –	107 –	113 –	120 –	127 –	135 –	142 –	150 –	160 –	169 –	180 –	190 –	203	frequency / %
C	C#	D	D#	E	E#	F#	G	G#	A	A#	B	B#	note-designation

Besides the ascending sequence of fifths, the descending sequence of fifths may also be generated: again neighboring notes are fifth-related. In correspondence to the example above, the initial frequency 100% would have to be repeatedly divided by 3/2: 67 %, 44 %, etc. With suitable octave shifts (towards higher frequencies), again a scale results – with calculated frequencies that slightly differ from the ones given above, though.

In the classical **Pythagorean tonal system**, not all of the notes calculated above were employed. Starting from the keynote C, users made do with 5 ascending fifths (C-G-D-A-E-B) and one descending fifth (F). They were able to form a **scale** that way:

1	$Q^2/2$	$Q^4/4$	$Q^{-1}\cdot 2$	Q	$Q^3/2$	$Q^5/4$	2
C	**D**	**E**	**F**	**G**	**A**	**B**	**C'**
1\1	8\9	64\81	3\4	2\3	16\27	128\243	1\2

In this table, Q represents the interval of the fifth♣ (frequency ratio 2\3); the corresponding exponent indicates the number of the required jumps of a fifth each. From the denominator, we can take the number of the additionally required octave shifts. $Q^5/4$ indicates 5 fifth-jumps towards higher frequencies, and subsequently 2 octave-shifts ($2^2 = 4$) towards lower frequencies. The third line yields, referenced to the keynote, the frequency relation as a fraction. The notes of the scale given above, and their frequency relation (interval), is designated according to their place number:
C = prime, D = second, E = third, F = fourth, G = fifth, A = sixth, H = seventh, C' = octave.

♣ To specify the frequency relations in an **interval-designation**, two different styles are customary: for the fifth e.g. 2:3 but also 3:2. Both relations are self-explanatory, while the letter-designation (C-G) does not unambiguously identify which one of the two is the lower note. In the following, the lower note is always positioned first (to the left) as is usual for axis-scaling. However, following through with this train of thought would result in fractions that are smaller than 1, such as e.g. $f_{C1} : f_{G1} = 2:3 = 0{,}666...$ While this representation is in itself correct, it is in contradiction with the practice of indicating intervals with number that are larger than 1. This contradiction is resolved in the following via using the back-slash (as used in Matlab): $f_{C1} \backslash f_{G1} = 2\backslash 3 = 1{,}5$.

The terms are related to numeration in Latin: primus, sekundus, tertius, quartus, etc. In its precise meaning according to the theory of harmony, these expressions designate the *distances* between two notes (inter-vallum = space between palisade beams), but in everyday use they also represent the names of notes: *the fourth on the C-scale is an F*. Distance in the above sense means to indicate the distance to the root note i.e. the ratio of the frequency of the note in question (e.g. an F) to the frequency of the keynote; in this example it is 3\4, corresponding to a fourth. It is also possible to form the ratio of two notes directly neighboring on the scale; this yields:

$f_C \backslash f_D = 8\backslash 9$; $f_D \backslash f_E = 8\backslash 9$; $f_E \backslash f_F = \mathrm{HT}$; $f_F \backslash f_G = 8\backslash 9$; $f_G \backslash f_A = 8\backslash 9$; $f_A \backslash f_B = 8\backslash 9$; $f_B \backslash f_{C'} = \mathrm{HT}$;

Of these 7 frequency ratios, 5 correspond to a so-called **whole-step** (‘whole note’, ‘whole tone’), specifically C-D, D-E, F-G, G-A, A-H. The remaining two intervals of neighboring notes are **half-steps** (HT, ‘half notes’, ‘semi-tones’, ‘half-tones’). In Pythagorean tuning, the frequency ratio in a whole step amounts to 8\9 = 1,125, and the one in a half step (E-F, B-C) **HT** = 243\256 = 1,053$_5$. The resulting scale is called **diatonic scale** because it is comprised of two different steps (namely whole-step and half-step). As supplemental information, ‘Pythagorean tuning’ should be indicated – there are many different tunings, after all.

N.B.: with respect to the **note that is internationally designated B**, there is a particular idiosyncrasy when the German language is used: there, this note is designated **H**. Originally (in fact: obviously), letters (starting with A) formed the names of the notes in the scale: A-B-C-D-E-F-G. However, medieval hexachord theory required (on top of the B as mentioned above) a second note half a step lower. In order to distinguish between the two, the designations *B-quadratum (B-durum)* and *B-rotundum (B-molle)* were introduced – derived from the angular (hard, *durum*) and round (soft, *molle*) writing styles of the letter *b*. The angular b mutated to an h … and now musicians in Germany, Austria, and the German speaking part of Switzerland found themselves with a peculiarity that continues to lead to (sometimes serious) complications when communicating internationally.

The diatonic scale as introduced above consists of 5 whole-steps and 2 half-steps. Each one of the whole-steps can pythagoreically be subdivided into two half-steps – however this may be done in two different ways. In the international note designations, half a step upwards is indicated with adding the syllable “sharp” to the note, and half a step downwards by adding the syllable “flat”. The diminished D is called **D-flat** (D*b*, with the *b* standing for ‘diminished’), the augmented C is **C-sharp** (C#). It has already been shown that all notes can be generated by using upwards-fifths and downwards-octaves in the Pythagorean sense:
C–G–D–A–E–B–F#–C#–G#–D#–A#–E#–B#.
However, all notes may just as well be generated via downward-fifths and upward-octaves:
C–F–B*b*–Eb–A*b*–D*b*–G*b*–C*b*–Fb–B*bb*–E*bb*–A*bb*–D*bb*.
The notes B*bb*, E*bb*, A*bb* and D*bb* result from diminishing B, E, A, D by *two* half-steps, respectively.

Fig. 8.3 shows the keynote frequencies of these two Pythagorean-chromatic scales. Due to the Pythagorean comma, no frequencies in a pair in the sequence of upwards-fifths and downwards-fifths are the same (except for the starting pair). If we limit ourselves to diminishing by a *single* half-step, a scale of 21 steps results: each of the 7 diatonic steps C-D-E-F-G-A-B is allocated a lower and a higher half-step. This 21-note tonal system was actually the basis for keyboard instruments – however it was deemed too complex.

Many musicians therefore simplified the scale by enharmonically equating similar notes. The resulting **12-step Pythagorean-chromatic scale** is indicated on the top of Fig. 8.3 via squares. Only a single half-step is introduced each between all whole-steps, but the half-tone distances are of different size, as is clearly visible (□—□).

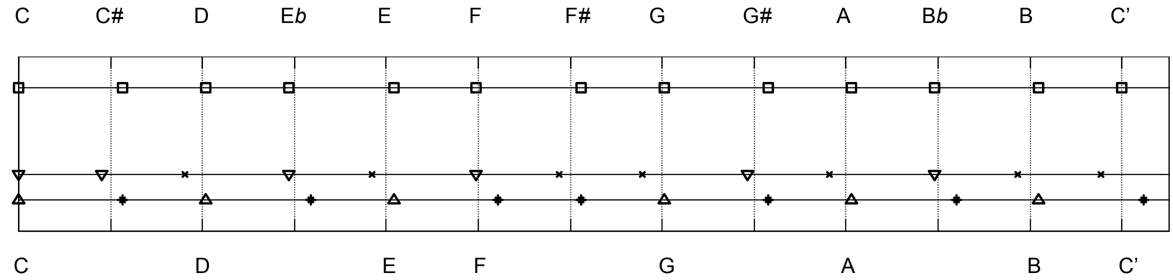

Fig. 8.3: Fundamental frequencies of the Pythagorean-chromatic scale, shown on a logarithmic frequency axis.
∆ = deduced from the first 6 upwards-fifths jumps; ∇ = deduced from the first 6 downwards-fifths jumps;
×,∗ = remaining 7 fifths jumps; □ = used in medieval times as chromatic scale.
The scale with equal temperament developed around 1700 is indicated with dashed vertical lines (8.1.3).

The different half-step distances complicate changing keys: the second (C-D) based on the keynote C has a larger frequency difference that the one based on C# (C#-E*b*), and other intervals (e.g. C-E, G#-C) meet a similar fate. Depending on the specific case, the flawed consonance when two notes are played simultaneously may be another problem. The fundamental thought behind the Pythagorean tuning was the note-relationship based on fifths and derived from the sequence of partials. Well meant that is – but you know how things are with relatives: as the distance grows, the similarities wane. **Fig. 8.4** schematically shows the frequencies of the partials for the prime (C) and the third (E). If, in simultaneous playing of the two notes, individual partials get to lie (frequency-wise) in immediate vicinity, **beats** may become audible. An example would be the 5th partial of the prime (C) and the 4th partial of the Pythagorean third (E_p).

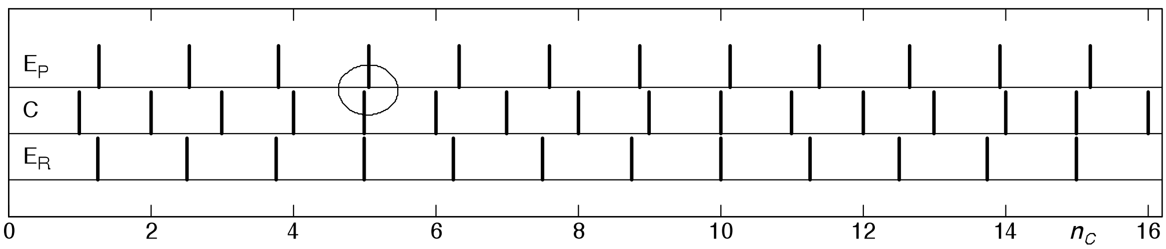

Fig. 8.4: Spectrum of partials of the notes C (prime) and E (third). Beats are generated between the 5th partial of the prime and the 4th partial of the Pythagorean third (E_P), due to the small frequency difference. For the pure third (E_R) the corresponding frequencies of the partial are identical. Abscissa: normalized frequency of the partials of the prime.

Beating happens when two mono-frequent notes of equal amplitude and similar frequency are played at the same time (i.e. they are added). Every note from a guitar consists of a multitude of (mono-frequent) partials, each of which is, individually considered, sine-shaped (a cosine-oscillation has the *shape* of a sine, as well). The 5th partial (= 4th overtone) of an ideal string vibrating at 100 Hz has the frequency of 100 Hz x 5 = 500 Hz, the 4th partial of the third according to Pythagorean tuning is at 126 Hz x 4 = 504 Hz. The frequency difference of the two partials is 4 Hz.

If we now regard merely the oscillation of the sum of the two partials, a figure similar to **Fig. 8.5** (center) results. The phase difference of the two partials fluctuates with the rhythm of the difference frequency, and amplification and cancellation alternate with the same rhythm. Given sufficient levels, *one single* partial with rhythmically fluctuating (i.e. beating) loudness is heard rather than two partials of almost equal pitch.

Interpreting the summation-curve (middle section of Fig. 8.5) is facilitated by reformulation towards a multiplicative operation:

$$\cos(2\pi f_1 t) + \cos(2\pi f_2 t) = 2 \cdot \cos(2\pi f_\Delta t) \cdot \cos(2\pi f_\Sigma t); \quad f_\Delta = \frac{f_2 - f_1}{2}; \quad f_\Sigma = \frac{f_2 + f_1}{2}$$

In this product-representation, f_Σ stands for the frequency of a cosine-oscillation with its amplitude changing "with the rhythm of the difference frequency f_Δ". The above example has f_Σ = 502 Hz, thus it lies exactly in between the primary frequencies f_1 und f_2. The term "difference frequency" should be used with care: it is calculated as f_Δ = 2 Hz, this is *half* the frequency distance between f_1 und f_2. However, the maximum of the beat-envelope appears (amount!) with double this frequency i.e. twice per f_Δ-period. The above beating with 500 Hz and 504 Hz as primary frequencies may therefore be seen as a tone at 502 Hz featuring 4 envelope maxima and 4 envelope minima per second. It therefore becomes louder and softer 4 times per second. The auditory effect of a beating of partials is difficult to predict – it may even be inaudible (despite its physical presence) due to masking by neighboring frequency components. If it indeed is audible, it may sound pleasant or displeasing. During many centuries the opinion was held that any beating of partials is undesirable, resulting in the beat-free **just intonation** (Chapter 8.1.2).

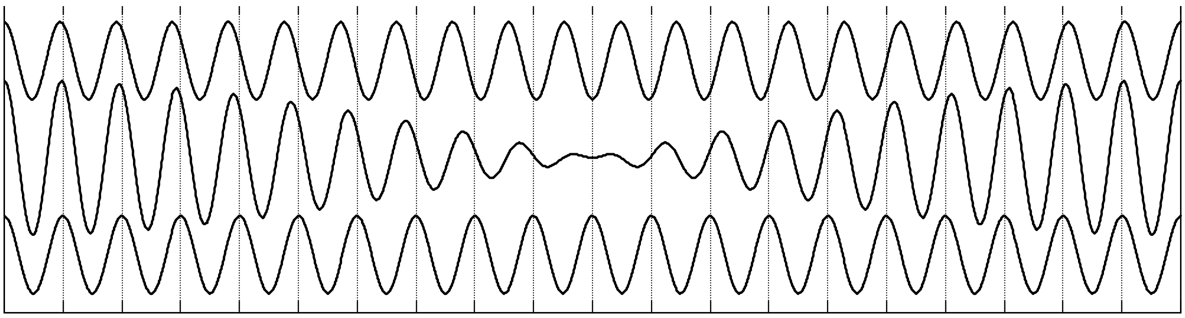

Fig. 8.5: Two cosine oscillations (top, bottom) slightly different (5%) in frequency, and their sum (middle). The curves are of equal phase at the left and right boundaries of the figure, and in opposite phase in the middle. Same-phase addition results in doubling of the amplitude (constructive interference), opposite-phase addition leads to cancellation (destructive interference). Abscissa: time.

8.1.2 Just intonation

In this context, *harmonic* and *natural* also stand as synonyms for *just* – the rationale being that nature herself allegedly had shown the way in the form of integer frequency ratios of the partials. The term *divine tuning* therefore is not far off, creating work for philosophers and esoterics, but mainly for mathematicians ... who not necessarily were musicians.

Just intonation – the term rings of *teachings of justice & purity*, and expressions such as *fairness*, *correctness*, or *well justified* come to mind – the opposite of *unjust*, *wrong*, or *unjustified*, and thus anything not conforming to the *just intonation* could in any case only be heresy. It is easily imaginable how hordes of mathematicians have deduced justifications for this or for that intonation … generating tables with an accuracy up to 12 figures! Or, rather: tables with 12 decimals, since the actual accuracy may have been a bit of an issue [Barbour]. Irrespective of any (not infrequently occurring) calculation- and rounding-errors: given a 1-m-long monochord string, 12-decimal-accuracy implies a length-tolerance of no more than 0,001 nm. Just to compare: the wavelength of visible light amounts to around 600 nm. Specifying pitch deviations with an "accuracy" of 1/10000000th of a cent is similarly nonsensical.

The **just intonation** may be traced back to ancient times. Two doctrines of thought emerged from the Pythagorean school (that originated around 530 BC): the **canons regular** (canon = rule, law) advocated the conservative opinion, while the **harmonists** gave priority to euphony, even if that required modification of mathematical laws of nature. The canonical doctrine regarded the frequency ratio 6 : 8 : 9 : 12 as "holy matrimony" between the forth and the fifth (**Fig. 8.6**) with the major second (full step F-G) being the result. Simbriger/Zehelein give an astounding assessment for this approach: *we have already met this grouping of notes in primitive music; with the Pythagoreans, we find that same basic occurrence substantiated and sanctioned with the background of advanced civilization.* There you have it: if – as a musician or listener – you recognize certain intervals as harmonic/consonant, then that's primitive ado. However, if you smudge some divine-cosmic-mystical mumbo-jumbo around that finding, it takes its place in high culture.

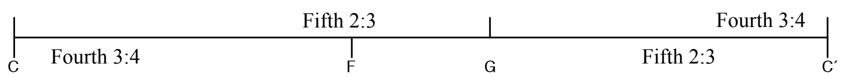

Fig. 8.6:
The "holy matrimony"

Still: despite some massive mystical sanctioning it was not possible to hide that the use of Pythagorean intonation made some chords sound less than pleasant. Young J.-apprentice: "oh honorable master Y.: them chords, they will not sound – try as I might! Those fifths and thirds, they fail to soothe us." Y.: "Do or do not: there is not try … but quiet now be, young one; in a special realm here taken we are. Let be it, for divine this is – of The Force" ☺. Many will have conformed to this sage advice from a long time ago and a galaxy far, far away … but some went public. In the olden days, on this planet, that could well lead to premature termination under artificially elevated ambient temperature – or it could open the door to eternal fame and glory. Or both. **Didymos** (Didymus) and **Ptolemy**, Alexandrinian savants by trade (and, to begin with, both by all means proponents of the Pythagorean third), evidently found the silver bullet (at the time probably the silver arrow). They replaced the Pythagorean third (based on the divine fifth) by an at-least-as-divine relation of whole steps: the major third – in Pythagorean intonation the frequency interval 64\81 = 1,2656 – was shifted to 4\5 = 1,2500 in the so-called Alexandrinian system. Didymos borrowed the minor third (27\32 = 1,1852) from the Pythagorean system, and Ptolemy modified it to 5\6 = 1,2000. In principle, anyway. Looking closer, we find [e.g. in Barbour] two didymian intonations, and no less than 7 ptolemyan intonations. Nevertheless, the foundation block for the just intonation was laid.

Studying literature, it is easy to come to the impression that (as mentioned above) something divine is connected to the just intonation. However, as confusion grows, the realization does manifest itself that it must in fact be a kind of polytheism. Barbour defines *just intonation* as: based on octave (1\2), fifth (2\3) and major third (4\5); the intervals themselves are designated *just* (or *pure*), as well.

Elsewhere, however, Barbour extends the term *just intonation* to: based on octave (1\2), fifth (2\3), fourth (3\4), major third (4\5), and minor third (5\6). Other authors even designate as *pure intervals* all intervals the frequency ratios of which correspond to the whole-numbered ratios of the frequencies of the first 16 partials. All intervals? Well, almost … those ratios that fit to some degree, anyway. But not the 7th , 11th , 13th , and 14th partials! Of course not. Valentin substantiates: *the miraculous, natural, and therefore not worked-out order of the whole system stems from the sequence of the composition of just intervals contained in these notes that – with a suitable octave transposition – yield our whole scale system.* The 7th, 11th, 13th, and 14th partials are the "black sheep"; nature finds space for something like that, too. Only for C-F# (or C-G*b*) no fitting frequency ratio at all could be found in the natural order. Therefore, the devil had to be called in as the usual suspect – only he/she could have smuggled in such an inconvenient, devilish interval (Tritonus, Diabolus in Musica). The question: "how could God allow this …" again created many workplaces for philosophers (compare Theodizee), but this would go beyond the scope of scientific considerations.

The just intonation derives its rationale from the whole-numbered frequency ratios of the first 16 partials. But why exactly 16 partials? That's because the 16th partial is exactly 4 octaves above the fundamental. But why then not just 1 or 2 or 3 octaves? That would be because that way you could not yet generate a chromatic scale. Moreover, wind instruments can just about reproduce the 16 "natural tones" (Eigen-tones, partials). The peculiarity of the tritone with its 45\64-ratio was justified on the basis of this fact that about 16 but not those 64 Eigen-tones could be generated. **Fig. 8.7** shows the frequency ratios of a just-intoned scale. Besides the devil's interval, there indeed is nothing fishy in there: numerators and denominators are integers between 1 and 16. The **major third** C-E that would with the Pythagorean intonation carry beats – it now is beat-free (compare to Fig. 8.4).

1 15\16 8\9 5\6 4\5 3\4 45\64 2\3 5\8 3\5 5\9 8\15 1\2
C – D*b* – D – E*b* – E – F – G*b* – G – A*b* – A – B*b* – B – C'

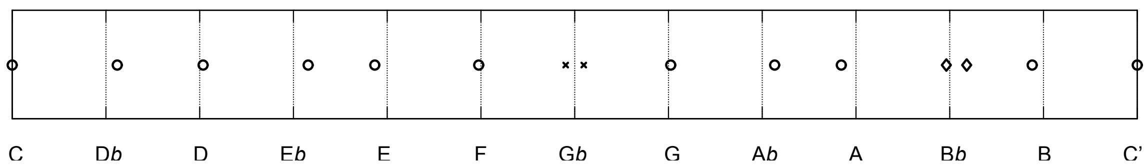

Fig. 8.7: Just intonation (Mersenne's lute tuning Nr. 2). The tritone was given also as F# with 32\45, for the Bb also 9\16 are found instead of 5\9.

Besides C-E, the combinations F-A and G-B (with 4\5) also make for a beat-free major-third interval. For the **minor thirds**, however, differences already appear now: E-G, A-C, and B-D yield 5\6, but D-F yields 27\32. Looking at the **fifth**-intervals: C-G, E-B, F-C, G-D and A-E yield 2\3, but D-A → 27\40. The **whole-step** intervals are at 8\9 or 9\10; the **half-step** intervals on the C-major scale are at 15\16, with the remaining (chromatic) half-step intervals at 24\25, 25\27 or 128\135. Despite the legitimization by nature herself, this gave opportunities for mockers: are you still learning, or do you play with a special intonation system?

It wasn't that these dissonances remained hidden to the working musicians. The latter knew about them, limited their music-making to a few keys, and tried to give a wide berth to the *howling-wolf intervals.* Alternatively, instruments could be built that divided every octave into 21 in-between notes. And if that didn't suffice: J. M. Barbour lists a plethora of other divisions, for example: the 31-division (Fibonacci-sequence), the 53-division (Bosanquet-harmonium), and don't you forget that *the 118-division has both fifths and thirds that are superlative (0,5 cent flat and 0,2 cent sharp, respectively)*.

There we are! … and there we go. On top of all that, other just (!) intonations were developed for the twelve-section octave, as well – which makes Barbour infer: **<u>the</u> just intonation does in fact not exist; rather, there are many different just intonations, with the best being the one that comes closest to the Pythagorean intonation**.

As desirable as "just" (or "pure") intervals may be in polychoral play: for intervals succeeding each other errors do cumulate. Take, for example (and see **Fig. 8.8**), Jimi Hendrix' "Hey Joe" (to solidly arrive back again in more modern times): the accompaniment first climbs down a minor third from E to C (or that could be interpreted as climbing up a minor sixth) and then runs through 4 jumps of fourths: E → C – G – D – A – E. Given just intervals, one revolution gets us this:

$$\frac{4\cdot3\cdot3\cdot3\cdot3}{5\cdot(2\cdot2)\cdot2\cdot(2\cdot2)\cdot2} = \frac{324}{320} = 1{,}0125 \mathrel{\hat{=}} +21{,}5\,\text{cent}\,;$$ C ↓ E = 5\4, fifth = 2\3, octave jump = 2\1.

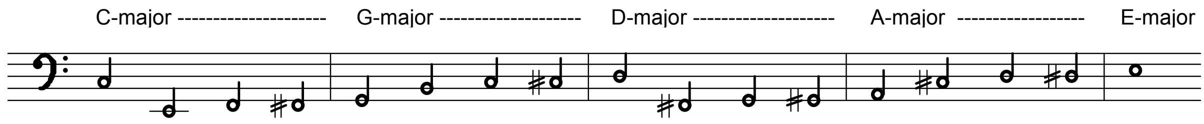

Fig. 8.8: Jimi Hendrix / Noel Redding: bass chromatic in "Hey Joe".

On the basis of just intervals, the full revolution of a cadence (lasting about 12 seconds in the original tempo) would lead to a frequency increase of 1,25% – after one minute, that would already make for no less that half a step. To execute every revolution at exactly equal pitch, e.g. the step from D to A (fifth) would have to be performed with the deviating ratio of 27\40. That, however, would mean a conflict with the pure (just) school of the first 16 natural notes.

Another "law of nature" (one that chalked up some success in architecture) is the **golden section** (or **ratio**). However, for Barbour the "golden tonal system of theoretical acoustics" is worth only a few lines. His bottom line: a jack-o'-lantern (ignis fatuus).

8.1.3 Tempered tunings

In music, the term temperament is used synonymously with the term tuning. **Tempered tuning** is no pleonasm, though. It is the technical term for tunings that, on a small scale and in a targeted manner, deviate from global tuning rules. Early versions of the tempered tuning may be traced back to Giovanni Maria Lanfranco (1533); starting from just intonation, he proposed to slightly down-tune the fifths, and to up-tune the thirds just as much as was tolerable (in terms of the perceived sound). During subsequent eras, there were countless attempts to define this advice more precisely. Starting from empirical results (the fifth should cause one beat per second), via graphical designs, nomograms, scary formula, and versatile tables, the path led to the equal-temperament tuning that dominates today: the octave is divided into 12 equidistant half-note-steps – and that's it. That this seemingly simple rule has not been in practice much longer – that is probably due to its demanding a readiness to compromise. It does require, after all, detuning those just and highly consonant intervals (such as the fifth). Not all musicians show a corresponding capacity for suffering: the cellist Pablo Casals speaks of the *brainwashing of the tempered tuning*, and the violinist Carl Flesch allegedly was unable to play together with a piano (in tempered tuning) subsequent to a rehearsal with a string quartet.

Well, with direct access to the string and thus the pitch as a continuum, a violinist has the advantage of a completely free-wheeling intonation. The piano does not offer this possibility. If a growth of the number of keys into the infinite is to be avoided, the only remaining solutions are a highly key-specific temperament, or a universal equal-beating (equal tempered) tuning.

Intervals between notes are characterized by the corresponding frequency ratios. Using the equal temperament, 12 similar half steps succeed one another within an octave, with a geometric frequency sequence resulting: 1, *HS*, *HS·HS*, *HS·HS·HS*, etc. Here, *HS* indicates the half-step interval, the 12-fold repetition of which yields the just (pure) octave: $HS^{12} = 2$. With this, the frequency relation of directly neighboring notes (at half a step distance each) calculates as:

$$HS = \sqrt[12]{2} = 1{,}059463...$$ Half-step interval in the equal temperament

The 12th root of 2 … that is an irrational number. In the actual sense of the word it is a number opposing reason. That may also be why a queasy feeling crept up on many a music-theorist. 3\4 for the just-intonated fourth is specified by nature herself; the counterpart in equal temperament, on the other hand, defies – with HS^5 – all sanity. And yet the numerical differences are not all that big: 3\4 = 1,33333... , HT^5 = 1,33484... , that's a gap of no more than merely 0,1%. However when principles are at stake, the gods themselves fight in vain. And sure: the differences may indeed be larger for other intervals. The following table lists all notes and frequency ratios in the equal-temperament scale. Other notes are not defined i.e. there is no distinction between C#/D*b*, E#/F, A*bb*/G, B/C*b*, and so on.

C	C#=D*b*	D	D#=E*b*	E	F	F#=G*b*	G	G#=A*b*	A	B*b*	B	C'
0	1	2	3	4	5	6	7	8	9	10	11	12
1	1,0595	1,1225	1,1892	1,2599	1,3348	1,4142	1,4983	1,5874	1,6818	1,7818	1,8877	2

Table: Notes and frequency ratios in equal-tempered tuning. The second line yields the half-note steps, the third yields the frequency ratios rounded to 4 decimal places. Reference = C.

In German-speaking lands, the term *gleichschwebend* (= with equal beating) could be misinterpreted such that all intervals would cause similar beating. This is not the case. The English designation EQUAL TEMPERAMENT is not self-explanatory, either. It is the half-note steps that are equal (in terms of the frequency ratios), and not the beats. Also equal (in the sense of relatively equal) is the distribution of the Pythagorean comma into all 12 jumps of fifths. Occasionally, *well-tempered* is found as a synonym for *equal tempered*; this can probably be traced back to **J. S. Bach's** preludes and fugues that he published under the title "The Well-Tempered Clavier". However, presumably Bach's instruments were not intonated with equal temperament (equal beats), but according to Werckmeister. Andreas **Werckmeister** (*Musikalische Temperatur*, 1691) had developed a tuning that comes close to the equal-temperament tuning but is not identical. Already one century earlier (around 1596), Simon **Stevin** had built a monochord the half-step frequency ratio of which corresponded to the 12th root of 2 (i.e. 1,059…). Presumably this was the first such instrument in Europe [Barbour]. Almost at the same time (around 1636), Marin **Mersenne**♣ carried out comprehensive theoretical groundwork.

♣ 1492 Franchinus Gafurius: Theorica musicae
1533 Giovanni Lanfranco: Scintille di Musica
1596 Simon Stevin: Monochord mit HT = $2^{1/12}$
1691 Andreas Werckmeister, Musicalische Temperatur
1511 Arnolt Schlick: Book on organ-building
1544 Michael Stifel: Arithmetica integra, z.B. log
1636 Marin Mersenne: Harmonie universelle
1706 Johann Neidhardt: Gleichschweb. Temp.

In his chapter *Equal Temperament*, Barbour lists no less than overall 41 different tempered tunings: eventual success had many parents that presumably had to fight vehemently for recognition. Even today, bitter adversaries turn up who are bothered by beating, “unnatural” intervals, while proponents of equal temperament revel in unlimited modulations. **Guitar players** should better make sure they run with the latter group because their instrument is manufactured using equal-temperament tuning.

In order to unambiguously define the whole relational range, it is also necessary to specify an **absolute value** besides just the frequency *relations* of the notes on a scale. As the long-standing reference (concert pitch), a^1 – the so-called middle A (also designated a' or A_4) – is in service. Today, the standard tuning frequency is **440 Hz** while in past centuries there were significant deviations in the range between 337 Hz and 567 Hz. In Germany, the reference was fixed to 422 Hz in Berlin in 1752. The year 1858 saw a proposal for international standardization on the conference on concert pitch in Paris, followed – on the corresponding conference in Vienna in 1885 – by the adoption of 435 Hz. On the ISA-conference in London in 1939, this value was increased to 440 Hz, and confirmed in 1971 by an ISO-resolution (ISO = International Standard Organization). In conjunction with the standardization, it was suggested to use the reference pitch for interval signals in radio and television, and as dial tone for the telephone. This was not a successful marketing idea: for the telephone, check measurements in 2004 showed a 6% deviation. The following table gives some fundamental frequencies for notes tuned to equal temperament; reference for A_4 is 440 Hz.

C	C#=D*b*	D	D#=E*b*	E	F	F#=G*b*	G	G#=A*b*	A	B*b*	B
523,25	554,37	587,33	622,25	659,26	698,46	739,99	783,99	830,61	880	932,33	987,77
261,63	277,18	293,66	311,13	**329,63**	349,23	369,99	392,00	415,30	440	466,16	493,88
130,81	138,59	**146,83**	155,56	164,81	174,61	185,00	**196,00**	207,65	220	233,08	**246,94**
-	-	-	-	**82,41**	87,31	92,50	98,00	103,83	**110**	116,54	123,47

Table: Frequencies of tones tuned to the equal-temperament scale, referenced to A_4 = 440 Hz; rounded to two decimal places. The open strings on the guitar E_2, A_2, D_3, G_3, B_3, E_4 are in bold.

In order to obtain convenient specifications of small deviations from correct tuning, Alexander John Ellis defined (in 1885) the **cent** as the (supposed) pitch-atom:

$$1\,\text{cent} = 2^{1/1200} = 1{,}0005778 \qquad\qquad \text{Interval} = 3986 \cdot \lg(f_2/f_1)\ \text{cent}$$

1 cent amounts to 1/100th of a half-step, or to the 1200th part of an octave. The frequencies 2000 Hz and 2001,155 Hz differ by 0,058% i.e. by 1 cent. Simbriger/Zehelein cite Preyer with the insight (questionable from a present-day perspective) that the hearing system was able to distinguish 1200 pitch steps between 500 Hz and 1000 Hz. Presumably, many a teacher scared away their pupils by demanding that the latter should be able to discern intonation errors of a 100th of a half-step. Chapter 8.2.2 has more on this topic.

8.1.4 Intervals in the equal temperament

The interval (inter vallum = space in between) is the distance of two notes; expressed numerically by the relation (ratio) of the frequencies of the corresponding tones. The names of the intervals are derived from the place numbers within the scale – for the C-major-scale, this implies: C = prime, D = second, E = third, F = fourth, G = fifth, A = sixth, B = seventh, C' = octave. Between the 3rd and 4th notes, and between the 7th and 8th notes, we find a half-step, all other notes are a whole-step apart each. In the equal-temperament tuning, a **whole-step** consists of two equal-size **half-step** (**HS**). All intervals can be represented by multiples of a HS:

Distance between notes (intervals) in the diatonic scale, represented by half-steps:
C-C = 0, C-D = 2, C-E = 4, C-F = 5, C-G = 7, C-A = 9, C-B = 11, C-C' = 12.

Intervals are not just definable as HS-multiples in their relation to the root note C of the C-scale, but also between all notes: e.g. D-E = 2 HS, G-H = 4 HS, F-A = 4 HS.

By the subdivision of the whole-step into two half-steps, new notes are obtained; they are designated by the chromatic sign relative to their neighbors: C# = C-augmented-by-one-HS, and (in the equal-temperament tuning) identical to the D*b* = D-diminished-by-one-HS. Corresponding: D# = E*b*, F# = G*b*, G# = A*b*, A# = B*b*. Equating the diminished notes and the augmented notes (e.g. C# = D*b*) is called the **enharmonic equivalent** (or enharmonic ambiguity). Out of experience, it appears that guitar players are more familiar with the augment-sign (#) than with the diminish-sign (*b*), and therefore we will give the former priority in the following. From the 7-step diatonic scale (C-D-E-F-G-A-B), a 12-step chromatic scale emerged:

C – C# – D – D# – E – F – F# – G – G# – A – A# – B chromatic scale

Each hyphen in this sequence represents a HS; the size of an interval can therefore be easily accounted for as HS-multiples. The regular numerals (second, third, fourth, fifth, etc.) are, however, already used (up) for the 7-step major (diatonic) scale, and this led to a somewhat confusing nomenclature: unison (0 HS, also called keynote or root), fourth (5 HS), fifth (7 HS) and octave (12 HS) are designated as **"perfect" intervals**, even if their tuning is not "pure" and free of beats! Caution is advised: C-G, for example, is designated a "perfect fifth" even in equal-temperament tuning. All other intervals within the major scale are "**major**" and thus: C-C = (perfect) unison, C-D= major second, C-E = major third, C-F = perfect fourth, C-G = perfect fifth, C-A = major sixth, C-H = major seventh, C-C' = perfect octave.

Reducing a large (major) interval by a HS results in a small (**minor**) interval. To get there, two possibilities exist: either the higher note is pushed down by a HS, or the lower note is pushed up by a HS: C-D*b* = C#-D = minor second, C-E*b* = C#-E = minor third, C-A*b* = C#-A = minor sixth, C-B = C#-H = minor seventh. If a perfect (or major) interval is enlarged by a HS we have an **augmented** interval; if a perfect (or major) interval is reduced by a HS we have a **diminished** interval. This results in two schemes:

diminished – minor – major – augmented (second, third, sixth, seventh)
diminished – perfect – augmented (unison, fourth, fifth, octave)

C-D# therefore represents an augmented second; in the sense of the enharmonic equivalent within the equal-temperament tuning, however, it also corresponds to the minor third C-E*b*. Purists turn away in horror, but the pragmatist just deals with it in everyday life: "C-D# is a minor third." Indeed, it is without purpose to ponder the differences between C# and D*b* when working with equal-temperament tuning. Of course, singers or violinists (as an example) will tend to intonate the augmented notes (#) slightly higher and the diminished notes (*b*) slightly lower, but that is then outside of equal-temperament tuning. When playing chords, the guitar player (and we are concerned with the associated instrument here, after all) has hardly any possibility to modify individual notes within the chord in their pitch. When playing single-note melody, higher-order knowledge of harmony could be put to use – unless the keyboard player in the band with his/her equal-temperament tuning shoots that down.

The following list gives an overview for **all intervals**, in this case referenced to C; with these representations: p = perfect, d = diminished, mi = minor, ma = major, a = augmented:

d-octave: C-C'*b*	p-octave: C-C'		a-octave: C-C'#
d-seventh: C-B*bb*	mi-seventh: C-B*b*	ma-seventh: C-B	a-seventh: C-B#
d-sixth: C-A*bb*	mi-sixth: C-A*b*	ma-sixth: C-A	a-sixth: C-A#
d-fifth: C-G*b*	p-fifth: C-G		a-fifth: C-G#
d-fourth: C-F*b*	p-fourth: C-F		a-fourth: C-F#
d-third: C-E*bb*	mi-third: C-E*b*	ma-third: C-E	a-third: C-E#
d-second: C-D*bb*	mi-second: C-D*b*	ma-second: C-D	a-second: C-D#
d-unison: C-C*b*	p-unison: C-C		a-unison: C-C#

This way, and given the enharmonic equivalent, every tone of the chromatic scale may exist in two different interval relationships to the keynote (in this case C):

C	perfect octave	12	augmented seventh	octave
B	major seventh	11	diminished octave	major-7th
B*b*	minor seventh	10	augmented sixth	seventh (mixo)
A	major sixth	9	diminished seventh	sixth (dorian)
G#	minor sixth	8	augmented fifth	#5
G	perfect fifth	7	diminished sixth	fifth
F#	augmented fourth	6	diminished fifth	tritone (lydian)
F	perfect fourth	5	augmented third	fourth
E	major third	4	diminished fourth	major 3rd
D#	minor third	3	augmented second	minor 3rd
D	major second	2	diminished third	whole step
C#	minor second	1	augmented unison	half step (phrygian)
C	perfect unison	0	diminished second	root

The *first* column in this table holds the designations of the note, the *second* column the preferred interval designations. The *third* column represents the half-note intervals relative to the keynote, and the *fourth* column represents the alternate designations. In the *fifth* column, some abbreviations customarily used by musicians are found (there may be others, of course). Again: this is based on equal-temperament tuning including enharmonic equivalents. Classical harmony theory finds reasons for a further differentiation; however, this is beyond the aim of the present elaborations [see secondary literature].

The below table indicates the numerical differences between just tuning and equal-temperament tuning. The deviation is just tuning vs. equal temperament tuning.

Interval name	no. of HS	notes	frequency relation	cents	deviation
Perfect octave	12	C-C'	1\2	1200,00	0,00
Major seventh	11	C-B	8\15	1088,27	-11,73
Minor seventh	10	C-B*b*	9\16	996,09	-3,91
Major sixth	9	C-A	3\5	884,36	-15,64
Minor sixth	8	C-G#	5\8	813,69	+13,69
Perfect fifth	7	C-G	2\3	701,96	+1,96
Tritone	6	C-#	32\45	590,22	-9,78
Perfect fourth	5	C-F	3\4	498,05	-1,95
Major third	4	C-E	4\5	386,31	-13,69
Minor third	3	C-D#	5\6	315,64	+15,64
Major second	2	C-D	8\9	203,91	+3,91
Minor second	1	C-C#	15\16	111,73	+11,73
Perfect unison	0	C-C	1\1	0,00	0,00

cent

Table: Frequency relations of octave-internal intervals for just tuning. The deviations refer to the corresponding interval in equal-temperament tuning. Specifications in 1/100th cents should be in practice rounded off to whole cent-values. Compared to the major third in equal-temperament tuning, the major third in just tuning is to low by 14 cents. The other way round: compared to the just-intonated major third, the major third in equal-temperament tuning is too high by 14 cents. A deviation of 1 cent corresponds to a frequency difference of 0,058%.

We can see the frequency relations for different tunings in the following **Fig. 8.9**. The abscissa is a logarithmically divided frequency axis.

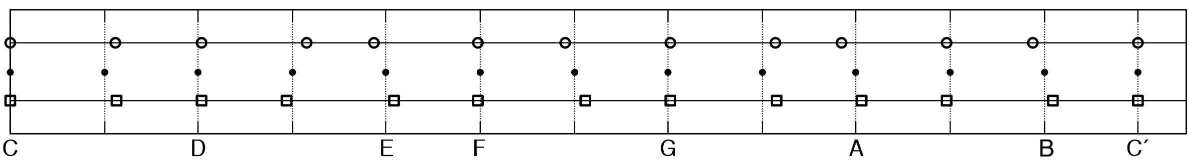

Fig. 8.9: Pythagorean (□), just-intonated (•), und perfect (o) intervals.

Since the half-step intervals are all equal in equal-temperament tuning, changing key (i.e. moving to a scale with a different keynote) does not represent a problem. For example, referencing to E results in the following intervals: E-F# = 2 HS = major second, E-A = 5 HS = perfect fourth. The reference to a particular key may now be omitted, because every interval is unambiguously defined by the number of its half-steps (HS).

Further interval designations exist **beyond the octave space**, as well: minor ninth (13 HS), major ninth (14 HS), minor tenth (15 HS), major tenth (16 HS), perfect eleventh (17 HS), augmented eleventh = diminished twelfth (18 HS), perfect twelfth (19 HS), minor thirteenth (20 HS), major thirteenth (21 HS); the half-step distances are given in brackets.

8.1.5 Typical detuning in guitars

Every guitarist will have experienced days when his/her guitar would just not tune properly. It typically gets really bad when we try to re-tune individual notes within *chords*. Even with a perfectly fretted neck and premium strings, this problem may occur – the most likely reason for which is the difference between just intonation and equal-temperament intonation. While the **fifth** tuned according to the latter is, with a deviation of 2 cents, really close to the perfect fifth tuned with just intonation, we find a much larger deviation for the **third**: that would be +13,7 cents for the major third, and as much as -15,6 cents for the minor third! Such detuning is already well audible, and the guitarist simply has to live with it. Trying to chord-specifically retune individual strings (towards just intonation) may easily generate deviations of as much as 29 cents for notes in other chords – and with that it now gets really shoddy. As an example:

An A-major chord (played without barré) consists of the notes [e-a-e-a-c#-e]. Given that all notes are tuned to equal temperament, it is in particular the C# played on the B-string that creates problems: it is sharp by 14 cents compared to a justly intonated C#. If we now retune the B-string by -14 cents (7,9 ‰), this A-chord will sound perfect. However: if, with the same retuning, e.g. an E-chord [e-h-e-g#-b-e] is played, the resulting sound is atrociously off. What happens is that the down-tuned B-string sounds a flat fifth – while the major third in that E-chord (the G# played on the neighboring G-string) is sharp. The interval between these two strings (3 half-steps G#-B) is too small by 29 cents! Changing from that re-adjusted A-major chord to a D-major chord creates a similar disaster: the down-tuned B-string now sounds too flat a D. The major third (F#) played on the neighboring E-string is already anyway too sharp by 13,7 cents and now sounds doubly out-of-tune relative to the tonic (that is lowered by 13,7 cents).

There may always be special cases when – given a limited selection of chords – a special detuning creates advantages. For example, it does not sound bad at all to slightly lower the tuning of the G-string for E and A7. E-major has [e-b-e-g#-b-e], and A7 has [e-a-e-g-c#-e]. In the E-major chord, the third profits, and in the A7 chord the diminished seventh – both are sharp in equal temperament relative to just tuning so that this detuning makes sense. For the same reason, the same detuning works well with the B7-chord [f#-b-d#-a-b-f#]. But don't you dare now changing to C or G … Thus, for universal deployment it is the equal-temperament tuning (executed as perfectly as possible) that remains a workable solution.

8.1.6 Stretched tuning

Piano tuners are known to tune not exactly according to equal temperament but in a slightly stretched-out fashion. In particular, in the very high and very low ranges, deviations of up to 30 cent can result. A spreading-out of partials, and in addition a narrowing of the pitch perception, are given as justification. In the guitar-relevant pitch range, however, the effect (merely 2 cents per octave) is rather weak, and the (compared to guitar strings) much heavier piano strings are no adequate equivalent. "Buzz" Feiten has obtained a US-patent for the stretched tuning – see Chapter 7.2.3). Fender, on the other hand, recommends adjusting the octave at the 12th fret with no more than 1 cent error – no spreading. To each his/her own …

