4.2 Magnetic Potentials

The magnetic field strength, as defined in chapter 4.1, is a differential length-specific quantity whose line integral yields the magnetomotive force. This figure can be interpreted as an integrated quantity (along the line) as well as the difference between the **scalar potentials** associated with the start and end points of the line. The potential defines the "magnetic power" of every point in space, whereas the magnetic field strength describes the spatial change of this "power". The word potential is derived from the Latin word "potentia" which means "ability, force, power, influence". The definition of a potential is also common in other areas, e.g. the gravitational field can be derived from the "potential energy". However, assigning an absolute power to every point in space within the framework of a relative scale immediately leads to the question about the **zero point** of this scale. In the case of temperature, there is an absolute zero deduced from energetic considerations. However, for the magnetic field this scaling is arbitrary. Strictly speaking the magnetic potential is not defined by a relative scale but rather by an **interval scale**, with zero being defined by a constant deliberately chosen for convenience. If one computes the magnetic field or the magnetomotive force as a potential *difference*, this constant will disappear. This leads to the legitimate question why a pseudo-absolute quantity (potential) is defined, if one continues to work with differences (intervals). The explanation can be found less in the area of physics but more in mathematics. The field and potential theory, which is based on complex function theory, offers a universal tool for the description of all fields, independent of their individual scaling.

In the **scalar potential** and the **vector potential**, mathematics provides us with two abstract quantities whose physical interpretation is somewhat arduous. First of all an obvious misinterpretation has to be ruled out: Even though the magnetic field is a vector field, it has both a vector potential and a scalar potential. The vector potential is a vector quantity associated to every point in space, the scalar product is a scalar quantity associated to every point in space. The scalar potential is, however, not the scalar value of the vector potential.

The **scalar potential ψ** is the quantity which leads to the magnetomotive force V through the formation of differences. If the distance between two points approaches zero, the respective potential difference converges towards the magnetic field strength $\vec{H}$. Hence, the differential quotient to be determined is the **gradient**:

$$\vec{H} = -\,\mathrm{grad}\psi = -\begin{pmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ \partial\psi/\partial z \end{pmatrix}$$

Magnetomotive force as a function of the scalar potential. The unit of the scalar potential is the Ampere.

The scalar potential ψ is, as suggested by its name, a scalar, the gradient is a vector. It points along the direction of the highest field *growth*. The field strength vector $\vec{H}$ points along the direction of the highest field *decrease* (*H*-decrease) since the equation contains a minus sign.

The gradient of a constant is zero. As the gradient formation is a linear operation, an offset does not change the gradient. As a consequence, the (arbitrary) definition of the potential zero has no influence on the field strength: grad(ψ) = grad(ψ + const).

It is easy to deduce the field strength from the scalar potential by calculating the gradient (spatial differentiation). Conversely, one has to calculate the line integral in order to deduce the scalar potential from the field strength. As always, integration needs an additive constant – the latter defines the absolute potential-zero. In the following equation this potential-zero is assigned to the point in space P_0. A line integral has to be formed between P and P_0:

$$\psi(P) = -\int_{P_0}^{P} \vec{H} \cdot d\vec{s} = \int_{P}^{P_0} \vec{H} \cdot d\vec{s}$$

Scalar potential as function of the magnetic field strength. $\psi(P)$ is the scalar potential at point P. At point P_0 the scalar potential is arbitrarily set to zero.

The **magnetic** scalar potential exhibits a specific feature: It is not **defined** universally and, where it is defined, it is either discontinuous or **ambiguous**. No scalar potential is allowed in sections of space where an electric current density $\vec{J} \neq 0$ is present. Inside an electric conductor no scalar potential exists. Not that it is zero, rather it is not defined. Outside the conductor a scalar potential can be defined, e.g. in the air, which is considered to be an insulator. If one defines the potential reference point [$\psi(P_0) = 0$] at a point P_0 outside of a straight current-carrying conductor and circles the conductor on a circular line, the potential will assume positive values. After a full circle one again arrives at P_0. The potential at this point equals the magnetomotive force. After two revolutions (arriving at the same point!) it amounts to twice the magnetomotive force. The scalar potential defined this way is continuous but ambiguous. Alternatively, one could restrict the definition range to one single full revolution. Then the scalar potential would become unique but would be discontinuous, because it changes its value abruptly at the borderline.

The second method is used frequently, i.e. a unique but (spatially) discontinuous scalar potential. For this, a sector or domain is defined in which an electrical current flow is not allowed (here $\vec{J} = 0$ is valid), and boundary lines are introduced so that this area will become **"simply connected"**. In a simply connected area *every* closed path may be reduced to a point. In **Fig. 4.4** the area outside the conductor is such a sector if the border line is introduced as a section boundary. It prevents a multiple circulation around the conductor, but at the same time produces a discontinuity (at the direct transition from C to A).

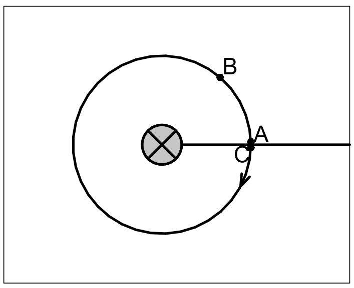

Fig. 4.4: Simply-connected area around a current carrying conductor. The line to the right is a sector boundary. The scalar potential will grow from A over B to C. The arrow indicates the direction of the *H*-vector.

It might be seen as disadvantage that the scalar potential is only defined outside the conductor. However, it does have the advantage that one (univariate) scalar is sufficient to describe of the field instead of the three field strength components (H_x, H_y, H_z) that would be otherwise necessary.

The **magnetic vector potential** is defined in addition to the magnetic scalar potential. It enables field descriptions inside as well as outside the conductor. However, the magnetic vector potential is not a very clear and accessible quantity. In fact, its existence is derived from formal mathematical considerations and subsequent numerical (FEM) calculations of the field (Potential and Field Theory, 4.9). The calculation of two-dimensional fields with the FEM-software "ANSYS" is only feasible with the vector potential and not with the scalar potential. The **vector potential** $\vec{A}$ is dependent on the field strength $\vec{H}$ via a special spatial differentiation, the rotation or curl:

$$\mu \cdot \vec{H} = \nabla \times \vec{A} = \operatorname{rot} \vec{A}$$

Vector potential[§] $\vec{A}$

Here μ is a material constant, the so-called permeability (chapter 4.3). In Cartesian coordinates the **rotation** is calculated as the difference of partial differentials and can be depicted with the nabla operator ∇:

$$\operatorname{rot} \vec{A} = \begin{pmatrix} \partial A_z / \partial y - \partial A_y / \partial z \\ \partial A_x / \partial z - \partial A_z / \partial x \\ \partial A_y / \partial x - \partial A_x / \partial y \end{pmatrix}$$

Curl in Cartesian coordinates.
The unit of the vector potential[§] is Vs/m.

For magnetic fields that can be represented by a **two-dimensional** scheme, e.g. parallel-plane fields, the vector potential has only one component. Both of the other components are zero. For example, the H_z-component is zero for an H-field only defined in the xy-plane. In the associated vector potential only A_z is non-zero. This is the component of the potential which is perpendicular to the xy-plane.

Fig. 4.4 represents such a parallel-plane field. The current flows into the plane of projection and an H-field emerges in the xy-plane. The vector potential has only an A_z-component parallel to the current flow. The equation specified simplifies to:

$$\mu \cdot \vec{H} = \nabla \times \vec{A} = \operatorname{rot} \vec{A} = \begin{pmatrix} + \partial A_z / \partial y \\ - \partial A_z / \partial x \end{pmatrix}$$

2D-vector potential

The vector potential presents an elegant method to define boundary conditions. This is necessary, for example, to reduce the complexity of computations or to set boundaries to infinite domains in FEM calculations. In addition, it is relatively simple to define field lines with the vector potential (chapter 4.7). **Figure 4.5** depicts the spatial vector relationship between the current density and field strength, and the flux density and vector potential, respectively.

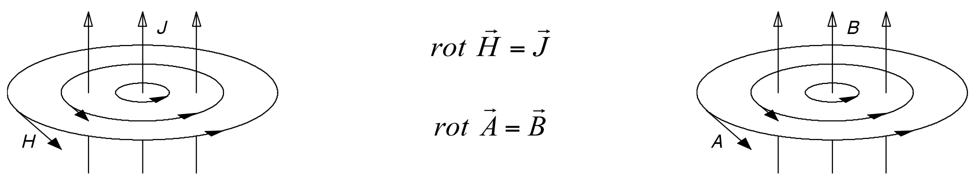

Fig. 4.5: Spatial relationship between $\vec{J}$ and $\vec{H}$ (left) and $\vec{B}$ and $\vec{A}$ (right).

[§] The symbol $\vec{A}$ must not be mixed up with the area vector!