4.9 Mathematical Field Theory

A field is defined as a special domain in which a physical quantity is depicted as a function of space and time. Mathematics supports the analytical field description by field and vector analysis as well as by complex function and transformation theories. The following gives a short overview of these theoretical field descriptions; detailed information can be drawn from the books of e.g. Bronstein, Papula, Smirnow, Heinold/Gaede.

The general field theory is powerful, but complicated. Simplifications are possible at the expense of generality which, in many cases, results in practically no restrictions with respect to precision. Magnetic fields propagate with the speed of light (~300000 km/s). The distances of approximately 10 cm relevant for pickups will, thus, be completed in 0.3 ns. Or the other way round: The **settling times** of the fields are much shorter than those that are typical for audio applications (µs – ms); this is why the time needed for the field set-up can be neglected. Simplifying, one assumes that the field change in the entire area takes place at the same time (phase-synchronous), that the magnetic field is free of memory effects, and that the current change of the field state is caused only by the actual excitation, not by spillover from the past. These kinds of fields are called quasi-static or quasi-stationary. A **quasi-static field** is in equilibrium and no energy transport or energy transfer is taking place. A **quasi-stationary field** is in a time-independent state of persistence, including energy transport or energy transfer. The prefix *quasi* stems from the fact that the time derivative $\partial/\partial t$ is *virtually* zero.

The magnetic field is described by both the vector field quantities $\vec{B}$ and $\vec{H}$. A vector field is called **conservative**, if the **rotation is zero** in the region under consideration; in this case the line integral is only dependent on the start and end points of the integration line and not on the integration path itself. Conservative fields are also called **potential fields**, because they possess a **scalar potential** ψ and the vector field quantity is the gradient of the scalar potential, which is the reason why it is also called a **gradient field**. The magnetic field is conservative only in the regions without electrical current flow. In addition, these regions have to be simply connected (Fig. 4.4). In mathematics a gradient field is often described by $\vec{x} = \mathrm{grad}\psi$, with $\vec{x}$ being the vector field quantity and ψ being the scalar potential. The gradient (of the scalar ψ) is a vector pointing into the direction of the largest field increase. However, for the magnetic field the respective formula contains a minus sign: $\vec{H} = -\mathrm{grad}\psi$. The field strength $\vec{H}$ of the magnetic field points into the direction of the largest potential decrease.

The spatial characteristics of the field vectors (e.g. $\vec{H}$) can be visualized by **field lines**. A curve $C(x,y,z)$ is a field line if the field vector at every point of the line is a tangent vector: $dx/H_x = dy/H_y = dz/H_z$. The solution of this differential equation yields the spatial vector field $\vec{H}(x,y,z)$. The field lines do not cross, except for the points at which $\vec{H}$ is not defined or becomes zero. For the magnetic field it can be useful to differentiate between field lines (originating from $\vec{H}$) and **flux lines** (originating from $\vec{B}$); flux lines sometimes are also called stream lines. Here, the literature does not have a consistent terminology.

The flux lines of a magnetic field are (as a rule) closed curves – without a beginning and an end. This is the origin of the expression **source-free field**. Obviously a natural impetus exists which is, however, not the origin of the flux or field lines. Within the framework of vector analysis, source-free means that the **divergence is zero**. Such a field sometimes also is called **solenoidal**. However, $\mathrm{div}\vec{H} = 0$ is not valid for every magnetic field; if ferromagnetics are incorporated, $\mathrm{div}\vec{H}$ can be non-zero because of the field-dependent permeability.

The expression flux line implies that something is flowing (streaming). **The flux** (the stream) can be objective (e.g. water circuit), or abstract (e.g. magnetic flux). As we just have explained, the field changes propagate with the speed of light, so it would appear obvious that the flow-rate corresponds to the speed of light. But one has to distinguish between the flow-rate and the velocity. If one throws a stone into the water, a circular wave propagates on the water surface. This, however, doesn't mean that all the water molecules move outwards with this velocity. The wave propagation velocity is much higher than the **particle velocity** which, for distinction, is called the *particle velocity*. The entire water surface would elevate and drop in phase for an infinite wave propagation velocity, without a visible wave pattern. The time differential of every particle displacement is the particle velocity – which also would be in phase across the entire water surface. The flux has the dimensions volume/time for flowing water whereas the area specific flux density has units of length/time, which might be interpreted as velocity. The dimension of the magnetic flux is Weber or Volt-second, the dimension of the flux density is T = Wb/m² = Vs/m². Here, it doesn't make sense to strenuously search for the dimension m/s. In fact, one might draw **analogous conclusions** on the basis of isomorphic (with identical structure) network graphs and the corresponding equation systems between magnetic circuits and water circuits (and other flow circuits). At the same time it should be clear, that the magnetic flux density does not correspond to the constant wave propagation velocity, but rather to the signal-dependent particle velocity. As the last analogy example we consider the **sound field**: the velocity of sound is constant at approximately 340 m/s. The velocity of the air particles is very much smaller – depending on the excitation. The sound flux q is derived from the particle velocity by multiplication with the area, not from the velocity of sound.

Within the framework of the above explained analogy considerations, $\vec{B}$ can be interpreted as a vector flux field, whose **flux integral** yields the flux Φ:

$$\Phi = \int_S \vec{B} \cdot d\vec{S} \qquad \Phi_q = \oint_S \vec{B} \cdot d\vec{S} = 0$$

Flux integral over S

The flux integral is the area integral over the area S. In case S is a closed, enveloping surface, as it is for the integral on the right, the flux integral describes the source flux emanating from the enveloping surface. As the magnetic flux lines do not have an origin and an end, it is clear that the source flux emanating from the volume limited by S must be zero; the field is source-free or solenoidal. If the envelope surface and the enclosed volume tend to zero, one obtains the divergence:

$$\mathrm{div}\vec{B} = \lim_{V \to 0} \left(\frac{1}{V} \oint_S \vec{B} \cdot d\vec{S} \right)$$

Divergence

The divergence is applied to vectors; the result is a scalar that provides information on the **solenoidal strength** at that position. Vector fields whose divergence is zero are called **solenoidal**. If one forms the line integral rather than the area integral on a vector field value, one ends up with the **contour integral**:

$$W = \int_s \vec{F} \cdot d\vec{s}$$ Work integral along the curve s

If $\vec{F}$ depicts a force, then the line integral results in the work performed (= force · path). Within the gravitational field the work integral only depends on the start and end points, not on the path taken. The value is zero in the gravitational field if the contour integral is computed over a closed line s (starting point = end point),. These fields are called **curl-free** fields. In analogy to the divergence, the curl (**the rotation**) can be depicted as the limiting case of an infinitesimally small circulation path:

$$\text{rot}\vec{H} = \lim_{S \to 0} \left(\frac{1}{S} \oint_s \vec{H} \cdot d\vec{s} \right)$$ Rotation (Curl)

Here we have introduced the magnetic field strength $\vec{H}$ instead of the force $\vec{F}$ whereby $\vec{H}$ is not necessarily curl free. s is the closed circulation path and S is the surface limited by s. Despite the possible confusion, the surface area is not depicted by the letter A as common in mathematics, because A or $\vec{A}$ denote the vector potential.

The following sentences of the field theory are given here without proof, full particulars can be found in the cited mathematics literature:

- A solenoidal vector field can always be depicted by the rotation of a vector potential.
- A curl free vector field can always be represented as the gradient of a scalar potential.

The relationships collected in **Fig. 4.33** can be understood most simply if one starts with the flux density. Magnetic fluxes are solenoidal because no magnetic monopole exists and, thus, the divergence is zero. This statement is depicted as Maxwell's Third Law, but sometimes also as Maxwell's Fourth Law – physicists do not have a common denomination: Since $\vec{B}$ is solenoidal one can always specify a corresponding vector potential $\vec{A}$. Using the permeability μ, one gets from the flux density $\vec{B}$ to the field strength, the line integral of which will yield the magnetomotive force V, which is coupled to the flux Φ through the magnetic conductivity Λ. The flux is the surface integral over the flux density $\vec{B}$. If one integrates the field strength over a closed loop s one will get the integrated magnetic field Θ. This equals the current I enclosed by s, which can be depicted as surface integral of the current density $\vec{J}$ over a surface S limited by s. The relationship between the field strength $\vec{H}$ and the current density $\vec{J}$ is described by Maxwell's First Law: its integral form equates the integrated magnetic field and the enclosed current (Ampere's Circuital Law). Its differential form connects the current density and the rotation of the field strength. In current-free areas the rotation of the field strength is zero and, thus, the field strength is curl-free and can, consequently, be interpreted as the gradient field of the scalar potential ψ.

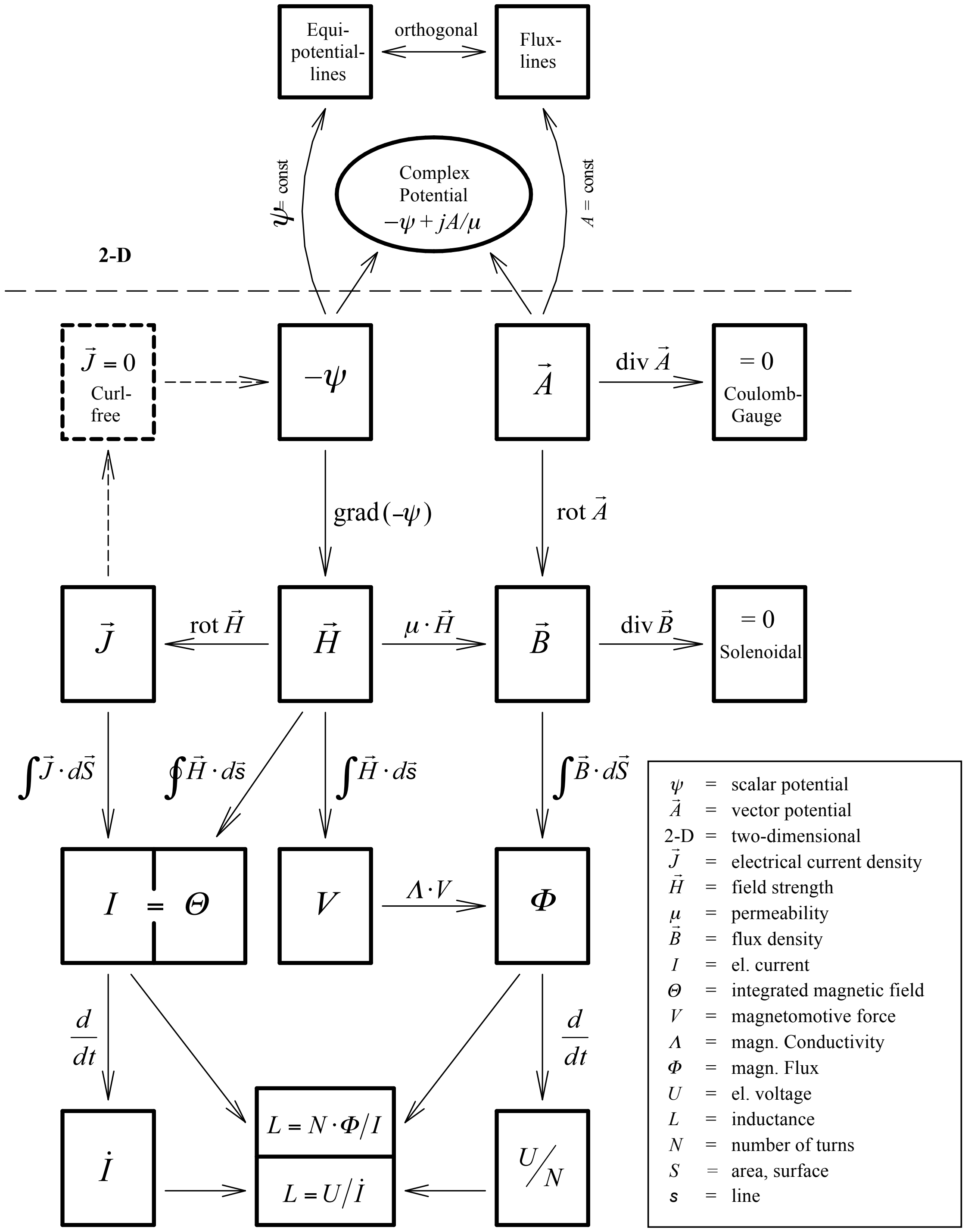

Fig. 4.33: Formal relationships between the magnetic field values. The complex potential is only defined for 2-dimensional fields. The scalar potential is only defined for current-free regions. This representation does not include wave propagation processes.

Under the assumption that μ is constant, i.e. that it is not position-dependent, $\vec{H}$ as well as $\vec{B}$ are curl-free and solenoidal. For the scalar potential this yields

$$\Delta\psi = \operatorname{div}(\operatorname{grad}(\psi)) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad \text{Laplace Equation}$$

This is the (homogenous) Laplace-Equation. If the Laplace-Operator Δ is applied to the vector potential instead of the scalar potential, the result is also zero:

$$\Delta\vec{A} = \operatorname{grad}(\operatorname{div}(\vec{A})) - \operatorname{rot}(\operatorname{rot}(\vec{A})) = 0\,; \qquad \operatorname{div}(\vec{A}) \overset{!}{=} 0 \qquad \text{Coulomb Gauge}$$

Basically, an integration from $\vec{B}$ to $\vec{A}$ offers an optional integration constant; it is chosen such that the divergence of $\vec{A}$ is zero (Coulomb-Gauge of the vector potential). Thus, the above given scalar Laplace-Equations are valid for every vector component of $\vec{A}$:

$$\Delta A_x = 0, \qquad \Delta A_y = 0, \qquad \Delta A_z = 0. \qquad \text{Vector Laplace Equation}$$

The Laplace-Equation is a very general linear homogenous differential equation. It can be applied to describe a solenoidal and curl-free field with only a single equation. However, within current carrying regions, the magnetic field is only solenoidal and not curl-free. Consequently, a scalar potential does not exist. For this application one employs the (inhomogeneous) Poisson-Equation:

$$\Delta\vec{A} = -\mu\cdot\vec{J} \longrightarrow \operatorname{rot}(\operatorname{rot}(\vec{A})) = \mu\cdot\vec{J}\,. \qquad \text{Poisson Equation}$$

The Laplace and Poisson equations assume a locally constant μ. However, in ferromagnetic materials the permeability is dependent on the field strength which is dependent on the position, yielding a location dependent permeability – the Laplace and Poisson equations are, consequently, not valid without restrictions.

A **complex potential** F can be defined in the **two-dimensional parallel plane** field. The formula symbols f or F are used in the mathematical literature; both characters are problematic because, in electrical engineering, f stands for frequency and F for force. In order to distinguish, F will subsequently be used. The components of the complex potential are **differentiable** complex functions, which are also called holomorphic, analytical or **regular**. Regular potential functions are invariant with respect to conformal mapping and, yielding simpler options for characterization and computation. The air flow around a complex airfoil can be projected onto two simple circles; the magnetic field around a cylinder can be mapped as a superposition of a parallel and a dipole field. Here, the orthogonality between equipotential and flux lines is conserved, because the conformal projection is isogonal. However, one may soon notice that the magnetic fields of pickups cannot be described sufficiently in two dimensions. The theory of the complex potential is only useful as a starting point to explain the basic relationships. The two cylinder axes are orthogonal for a cylinder magnet and a string; this situation cannot be described by parallel plane or by rotational symmetry. Rather, a 3-dimensional coordinate system is necessary – and the complex potential is not defined within this boundary condition.

The point variable of the complex potential is $z = x + jy$. Here x and y are the abscissa and the ordinate of a two-dimensional coordinate system, respectively. The scalar potential ψ is a regular potential function of z, for which the differentiability and the regularity are given by the applicability of the Laplace differential equation. The scalar potential is then considered to be the real part of the regular complex function F or, in other words, the real part function ψ is supplemented by an imaginary part to become an **analytical function**. This imaginary part is defined by ψ in a unique way, because F should not become any complex function but a regular (=analytical) one. The Cauchy/Riemann differential equations (**C/R-DGL**) are valid for regular functions and, as a result, an imaginary part can be derived for every real part. As for every integration, an additive constant can be freely chosen. It is determined by the Coulomb Gauge.

The definition of the complex potential is, however, yet not entirely **unique**: One could consider ψ also as imaginary part to which a real part is supplemented and also the sign can be arbitrarily assigned. In general notation the complex potential is:

$$F(z) = u(z) + j\,v(z)$$ Complex potential

The C/R-DGL have to be valid because F should be a regular function in the region under consideration:

$$\partial u/\partial x = \partial v/\partial y, \qquad \partial v/\partial x = -\,\partial u/\partial y$$ Cauchy/Riemann

Mathematics interprets u to be the scalar potential of a curl-free vector field, which can be depicted as the gradient of u. The gradient is a vector which points in the direction of the largest field increase. However, **physics** defines the magnetic scalar potential ψ (also the electrical scalar potential φ) as a vector pointing in the largest field decrease. It is, therefore, obvious to assign a minus-sign to the real part of F:

$$F(z) = -\psi(z) + j\,v(z)$$ Sign facultative

The gradient of the real part of F is the field strength vector, the components of which can be translated into the imaginary part of the complex potential with the help of C/R-DGL:

$$\vec{H} = -\text{grad}\psi = \begin{pmatrix} -\,\partial\psi/\partial x \\ -\,\partial\psi/\partial y \end{pmatrix} = \begin{pmatrix} \partial v/\partial y \\ -\,\partial v/\partial x \end{pmatrix}$$ ψ = scalar potential

A corresponding relationship can be derived for the vector potential, whose rotation yields the flux density. In two dimensions (for which the current description is valid) the **vector potential** has only one component $A = A_z$. The index z here refers to the third Cartesian coordinate of the x-y-z-system; it should not be mixed up with the complex space coordinate $z = x + jy$ of the two-dimensional field!

$$\vec{H} = \frac{1}{\mu}\vec{B} = \frac{1}{\mu}\text{rot}\vec{A} = \frac{1}{\mu}\begin{pmatrix} \partial A/\partial y \\ -\,\partial A/\partial x \end{pmatrix}, \quad v = A/\mu, \quad \mu \neq \mu(z).$$ $\vec{A}$ = vector potential

It follows for the **complex potential** of the magnetic field:

$$F(z) = -\psi(z) + j\,A(z)/\mu$$ Complex potential

The complex potential is a combination of the scalar potential ψ and the vector potential $\vec{A}$, which are, in turn, functions of the field strength. The rotation of the field strength is zero in the **curl-free** magnetic field:

$$\mathrm{rot}\vec{H} = 0 \longrightarrow \partial H_y/\partial x - \partial H_x/\partial y = 0 \longrightarrow \partial H_y/\partial x = \partial H_x/\partial y$$

This is the so-called *integration condition* of a plane vector field, a necessary and sufficient condition for the independence of the line integral on the integration path and for the complete differential $d\psi$:

$$-d\psi = -\frac{\partial\psi}{\partial x}dx - \frac{\partial\psi}{\partial y}dy = H_x \cdot dx + H_y \cdot dy \qquad \text{Complete differential}$$

For $d\psi = 0$ one obtains curves of constant scalar potential ψ = const., the so-called equipotential lines. The slope *dy/dx* of these lines correlates with $-H_x/H_y$, i.e. the equipotential lines are normal to the direction of the field strength vector.

In a **solenoidal** magnetic field the divergence of the flux density is zero and, hence, the divergence of the field strength is zero for a location independent μ:

$$\mathrm{div}\vec{H} = 0 \longrightarrow \partial H_x/\partial x + \partial H_y/\partial y = 0 \qquad \text{for } \mu = const\text{, i.e. } \mu \neq \mu(z)$$

The complete differential *dA* of the vector potential *A* leads to:

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = -\mu H_y \cdot dx + \mu H_x \cdot dy \qquad \text{with } \mu\vec{H} = \mathrm{rot}\vec{A}$$

One gets curves of equal vector potential (A = const.) for $dA = 0$, whose slope *dy/dx* corresponds to the slope of the field strength vector: $dy/dx = H_y/H_x$.

Hence, curves of equal vector potential form the directional field of the field lines, with curves of equal scalar potential (equipotential lines) being normal to them.

Since the complex potential F is defined as a regular (analytical) function, every regular projection of F must result in a regular function of z. The complex derivative *d/dz* is such a regular projection (satisfying C/R-DGL); if applied on the complex potential this yields:

$$dF(z)/dz = \frac{\partial}{\partial x}\mathrm{Re}\{F\} - j\frac{\partial}{\partial y}\mathrm{Re}\{F\} = -\frac{\partial\psi}{\partial x} + j\frac{\partial\psi}{\partial y} = H_x - jH_y = H^*$$

The derivative of the complex potential corresponds to the conjugated complex field strength, whose x and y components are interpreted as the real and imaginary parts. H^* is also a regular function of z. The complex integral over the conjugated field strength is the complex potential. The additive constants for ψ and A are arbitrary for this integration; they define the origin of the scalar and vector potentials.