## 5.8 Non-linear Distortion

Both the measurements with the motorized test bench (chapter 5.4.4) and the measurements with the shaker (Fig. 5.4.23) make us surmise that the functions of distance relationships are in fact (non-linear) power functions – in the framework of a diverging magnetic field this would not be surprising. However, sinusoidal excitations fed into non-linear functions will lead to non-linear distortion i.e. to the generation of new frequencies. Such a system can be seen as linear only for very small (string) excursions; the amplitudes occurring in the practical musical application are rather strong such that the large-signal-behavior needs to be investigated, as well.

In order to explain the basic relations, let us first look at a system with a transfer characteristic including a linear term and a square term:

$$y(t) = a \cdot x(t) + b \cdot x^2(t); \qquad x(t) = \hat{x} \cdot \sin(\omega t)$$ Transfer characteristic; signal

The squared sine-function can be seen as a superposition of a (constant) DC-component and an oscillation at double the frequency:

$$y(t) = a \cdot \hat{x} \cdot \sin(\omega t) + b \cdot \hat{x}^2 \cdot \tfrac{1}{2}\big(1 - \cos(2\omega t)\big)$$ Nonlinear distorted signal

The spectral representation of $y(t)$ shows three components: the DC-part at 0 Hz, the first harmonic at $\omega$ and the second harmonic at $2\omega$. Form this we obtain the 2nd-order harmonic distortion $k_2$ at:

$$k_2 = \frac{b \cdot \hat{x}^2}{\hat{x} \cdot \sqrt{4a^2 + b^2 \cdot \hat{x}^2}} \approx \frac{b \cdot \hat{x}}{2a}$$ 2nd-order harmonic distortion

A value used often instead of harmonic distortion is the (2nd-order) **harmonic distortion attenuation**:

$$a_{k2} = 20 \cdot \lg\left(1/k_2\right) \mathrm{dB} \approx L_1 - L_2$$ Harmonic distortion attenuation

$L_1$ is the level of the 1st harmonic and $L_2$ the level of the 2nd harmonic. The approximation holds, strictly speaking, only for small signal levels but will be used without constraints in the following.

In the **general case** the transmission curve does not only include a 2nd-order distortion component but further series components of higher order:

$$y(t) = a \cdot x(t) + b \cdot x^2(t) + c \cdot x^3(t) + \ldots$$ General transmission characteristic

Any continuous function can be expanded into such a series (Taylor-MacLaurin). The corresponding spectral representation includes not only the additional 2nd order harmonic but also further lines (higher harmonics) at integer multiples of the fundamental frequency. The distortion components in power function decrease with the order and therefore we will regard only the dominating 2nd-order distortion as a simplification.

If a nonlinear system is excited not with a mono-frequency signal but with a **mixture of frequencies**, not only multiples of the fundamental frequencies result but also summation and difference frequencies. For the ideal, **dispersion-free** string, exactly harmonic partial tones are assumed, i.e. for example 100, 200, 300, 400 Hz. The difference tones generated by the non-linearity (as described above) correspond exactly to already existing frequencies. The flexural rigidity of real strings, however, generates dispersion and a frequency-spreading resulting in complicated spectra. For every primary tone (e.g. 100, 201, 302.3, 404 Hz), neighboring lines come into being which lead to additional beat-like **modulations**.

The typical transmission curve shown in **Fig. 5.8.1** describes the correspondence between the distance of string to magnetic pole and the magnetic flux. With the string in the still position, the distance between magnetic pole and string is $d$ = 2 mm in this example (operating point). A sinusoid movement of the string with an amplitude of 1,5 mm leads to a non-linear flux change, in which the negative half-waves have smaller value that than the positive half-waves. The induced voltage is proportional to the flux *change* over time (law of induction, $d\Phi / dt$), and a saw-tooth like curve results for the voltage. In this example the square harmonic distortion attenuation is about 12 dB, corresponding to a 2$^{nd}$ harmonic distortion of about 25%. The 3$^{rd}$-order harmonic distortion attenuation amounts to about 26 dB ($k_3$ = 0,5 %).

The square harmonic distortion is approximately proportional to the amplitude of the string vibration. For the above example and an excursion of 0,5 mm, $k_2$ decreases to about 8 %, and $k_3$ to about 0,055 %.

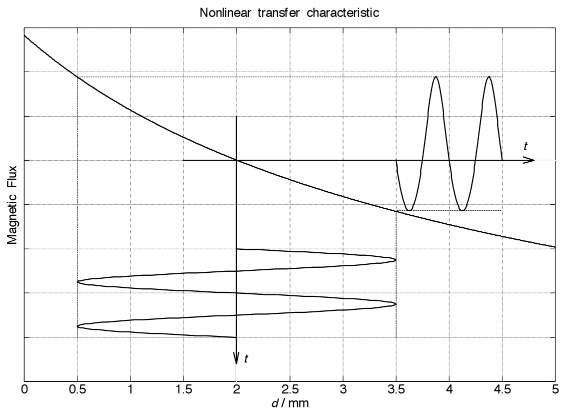
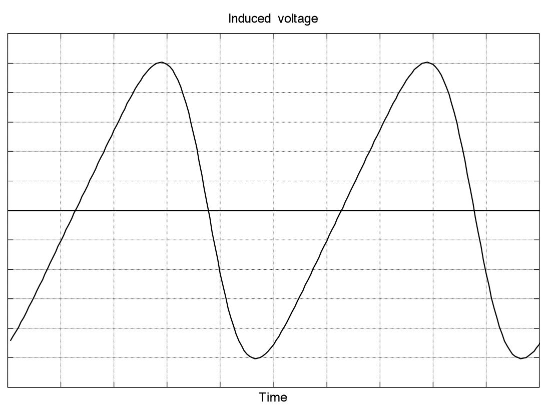
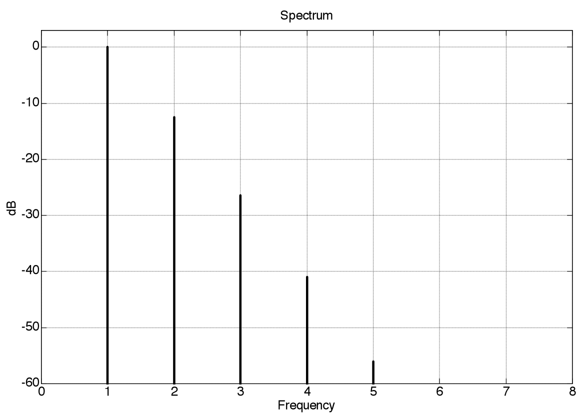



**Fig. 5.8.1:** Curved transmission characteristic and sinusoid excitation (left). The time-derivative of the distorted magnetic flux has a saw-tooth shape (lower left, the spectrum contains all whole-numbered harmonics (below).



The following measurements were taken on the shaker test bench at 84 Hz. As was established with an acceleration sensor, initially the shaker itself had a harmonic distortion of $k_2$ = 2%. This value could be improved to 0,1% via compensation – a base line which is more than adequate in view of the much higher pickup distortions. In Fig. 5.8.2 the results for singlecoil and humbucking pickups are shown. The string excursion was 0,4 mm for all measurements, the clear span (distance *d*) between the (still) string and the magnetic pole was varied between 0,5 mm and 5 mm.

**Fig 5.8.2:** Harm. distortion attenuation $a_{k2}$, $f$ = 84 Hz, excursion amplit. = 0,4 mm. String diameter = 0,66 mm. Abscissa: distance string/magnetic pole *d*. For the T-Iommi-pickup, the distance string/cover is used, the distance to the magnetic pole is thus larger, and the curve needs to be shifter right for comparison.

For all pickups the distortion decreases with increasing distance; within the relevant range of *d* the 2nd order harmonic distortion amounts to 4 – 5% for 0,4 mm string excursion. Considering that with strong picking 2 mm excursion can easily be reached, a harmonic distortion of above 10% is possible. This is, however, not a characteristic of a special pickup but occurs similarly in all investigated pickups. As with comb-filter responses, it is necessary to take into account that every pickup is part of a musical instrument: one can objectively describe its transmission characteristic but an evaluation remains a subjective affair. Since the vibration of each string is distorted individually (without interaction with neighboring strings, see below), the effect of the distortion is much less spectacular than the numbers would appear to indicate. Clearly audible distortion is generated mainly in the electronics to which the pickup is connected but not in the pickup itself.

**Fig. 5.8.3** compares measurements and calculations. As a good approximation, the field transmission characteristic of a **Stratocaster** pickup follows a simple **power function**:

$$\Phi = K_0 + K_1 \cdot \left(\Delta + d + x(t)\right)^{-1} \qquad \Delta = 4{,}3\,\mathrm{mm} \qquad \text{Field-transmission characteristic}$$

The levels of the first and second harmonics dependent on the distance *d* (left) and the excursion amplitude $\hat{x}$ (right) agree very well with the measurements. The static magnetic flux (no string excursion) can be defined via the constant $K_0$; for AC-considerations its value is without importance since it disappears in the process of differentiation. The constant $K_1$ determines the transmission coefficient. For small string excursions it is (for 7600 turns on the coil) $K_1 = 1{,}1 \cdot 10^{-9}$ Vsm . Taking the magnet cross-section as the area through which the field penetrates yields – with $d$ = 2 mm and $\hat{x} = 0{,}4\,\mathrm{mm}$ – a flux-density amplitude of 0,5 mT.

This is only a coarse estimate since the magnetic flux is not concentrated on the magnet cross-section but spreads into neighboring areas. The area is therefore larger than assumed. At the same time, it is necessary to consider that not all turns of the winding are penetrated by this magnetic flux. The number of turns therefore is smaller than assumed. These two errors are conveniently opposed and the overall estimate should not be too far off. Compared to the DC-component of the flux density (which amounts to about 100 mT at the end face of the magnet) the AC component is very small for the parameters as given above, and a linearization for the calculation of the fundamental oscillation is possible without large errors. The nonlinear behavior is described by the given characteristic with sufficient accuracy.

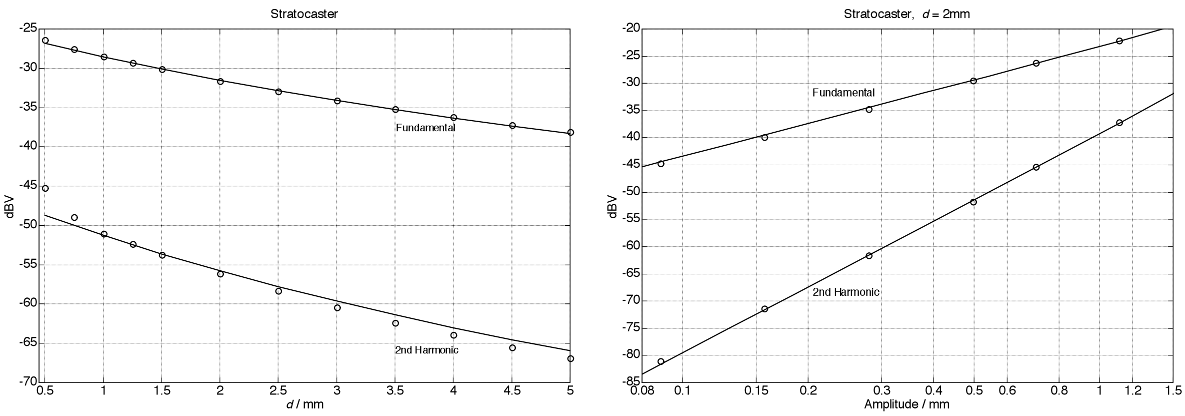



**Fig. 5.8.3:** Dependency of the level of the 1st and 2nd harmonic on the distance (left) and the amplitude (right). The dots are measured values, the lines result from a calculation of the power function as discussed in the text.

The real string vibration does not include only a single frequency but is a **frequency mixture** from many partials. If all these frequencies were exact multiples of the fundamental, the non-linearities of the pickup would create new components exclusively at the already present frequencies. For example, the $3^{rd}$-order distortion of the fundamental generates (amongst other components) a tone at three times the fundamental frequency – i.e. exactly at the frequency of the $3^{rd}$ partial ($3^{rd}$ harmonic). However, the partials of the string are **not exactly harmonic**: flexural rigidity, magnetic filed of the pickups and frequency-dependent bearing impedances lead to a spreading of the frequencies of the partials. An $E_2$-string tuned to exactly 82 Hz could e.g. have a $3^{rd}$ harmonic which is shifted from 246,0 Hz to 246,3 Hz. A $3^{rd}$-order distortion of the fundamental will create (amongst other components) a distortion product at 3 · 82 Hz = 246,0 Hz which creates a beat-like amplitude change with the 3rd harmonic (246,3 Hz). Since, however, every string vibration in reality includes modulation of the partials anyway, the additional modulation generated by the pickup is insignificant.

Completely immaterial are **nonlinear string interactions**. The primary tones ($f_1, f_2$) generated by two strings interact and produce sum- and difference-tones ($n \cdot f_1 \pm m \cdot f_2$) due to the nonlinear characteristics. To measure this effect quantitatively, 2 neighboring strings were strongly plucked and the pickup output voltage was analyzed. This was done for the following pickups: Gibson '57-Classic, Gibson Tony Iommi, DiMarzio DP184, Fender Texas-Special-Telecaster. Even with a mere 1-mm-distance between string and magnet, the intermodulation remained below 0,1%. The main reason for pickup distortion is the magnetic resistance of the field in air between string and magnet – this resistance being nonlinearly dependent on the string position. The neighboring string vibrating at a distance of about 1 cm has practically no influence on this process.

The magnetic flux changes generated by individual strings do superimpose in the magnet (or in the field-shaping pole pieces) – but the relative flux changes are so small that the – in principle non-linear – hysteresis may be linearized, after all. String interactions and string intermodulation starts to play a role only as non-linear distortions appear in the amplifier.

On top of the interactions resulting from two strings, the term **intermodulation** could however also be considered regarding the combination tones generated by individual partials of <u>one</u> string. Strong low-frequency string excursions shift the operating point on the non-linear transmission characteristic (Fig. 5.8.1), and as a consequence the amplitude of the higher frequency partial changes. Again, the shaker test bench delivers quantitative data: a D'Addario string (0,66 mm diameter, PL026) was adjusted to 2 mm distance to the magnetic pole. A low-frequency vibration (20 Hz, 0,55 mm amplitude) was added to a higher-frequency vibration (80 Hz, 0,23 mm amplitude), with the vibrations oriented in parallel to axis of the magnet. As a result of the non-linearity, new spectral components appear with the 60-Hz- and 100 Hz-lines being of particular interest. In the idealized **model**, the two-tone-mixture $x(t) = \cos(\omega t) + k \cdot \cos(\Omega t)$ receives a 2nd-order distortion: $y(t) = \kappa \cdot x(t) + x^2(t)$.

$$y(t) = \kappa \cdot x(t) + \cos^2(\omega t) \; + \; k^2 \cdot \cos^2(\Omega t) \; + \; 2k \cdot \cos(\omega t) \cdot \cos(\Omega t). \qquad \text{Non-linearity}$$

With $\cos^2\alpha = (1 + \cos 2\alpha)/2$ and $\cos\alpha \cdot \cos\beta = [\cos(\alpha + \beta) + \cos(\alpha - \beta)]/2$, the new frequencies resulting from the non-linearity can be easily calculated: next to the DC component (0 Hz, unimportant in this context), the double primary frequencies ($2\omega$, $2\Omega$) and the sum- and difference-frequencies ($\Omega + \omega$, $\Omega - \omega$) occur. The 3-tone-mixture of $\Omega - \omega$, $\Omega$ and $\Omega + \omega$ can be interpreted as classical amplitude modulation [e.g. 3]. A more descriptive approach: the low-frequency primary tone (20 Hz in the example) shifts the operating point back and forth along on the curved (non-linear) characteristic, and the additionally present higher-frequency signal (80Hz) finds a time-dependent steepness of the characteristic curve. In the ranges of higher steepness, the output signal is stronger, and for lower steepness correspondingly weak (**Fig. 5.8.4**). For the measurement, the 20-Hz-amplitude was 0,55 mm, and the 80-Hz-amplitude amounted to 0,23 mm. For the calculation to be compared to the measurement, the same characteristic as in Fig. 5.8.3 was used, and the correspondence is acceptable (Fig. 8.5.4, left). The harmonic-distortion model therefore fits also well for describing intermodulation distortions.

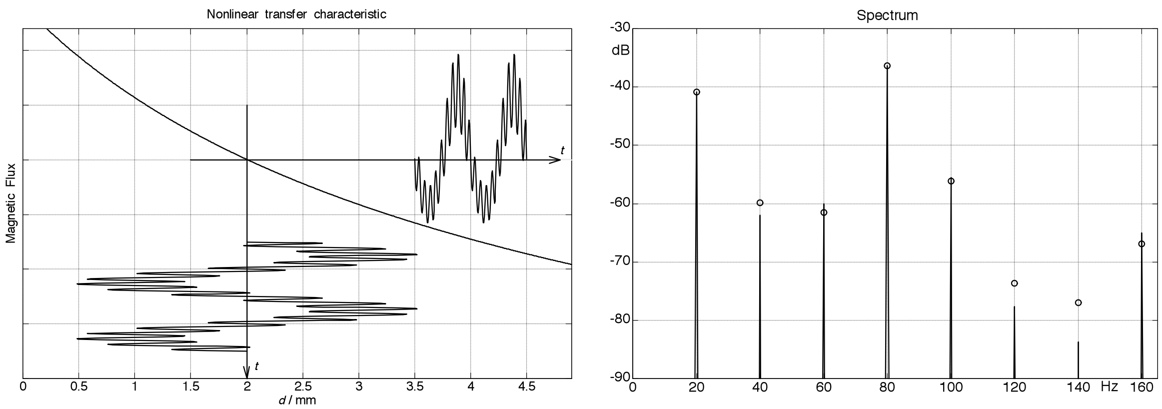



**Fig. 5.8.4**: curved characteristic with two-tone signal (left). The right-hand section shows measurements (o) and the correspondingly calculated string-velocity-spectrum; characteristic as in Fig. 5.8.3. The shape of the curve in the left section shows the basic relations but does not correspond to the data of the right-hand figure.