## 6.4 Reciprocity

The theory of two-port networks describes the relationships between the input- and output-quantities of linear, time-independent systems using **two-port matrices**. For the electrical two-port system, the input signals applied to the input connectors (port 1) are input voltage $U_1$ and input current $I_1$; correspondingly, output voltage $U_2$ and output current $I_2$ are found at the output connectors. Connecting these four quantities, **two-port equations** may be defined:

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \qquad \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \cdot \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

Impedance matrix $\boldsymbol{Z}$
Admittance matrix $\boldsymbol{Y}$

$$\begin{pmatrix} U_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ U_2 \end{pmatrix} \qquad \begin{pmatrix} I_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \cdot \begin{pmatrix} U_1 \\ I_2 \end{pmatrix}$$

Hybrid matrix $\boldsymbol{H}$
Inverse hybrid matrix $\boldsymbol{G}$

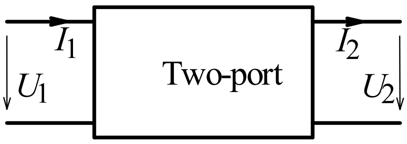



**Fig. 6.9:** Electrical two-port with technical directions of the reference arrows. In the general case, all quantities are complex.

Besides these four matrices, we may also define the chain matrix (***A***) and the inverse chain matrix (***B***); theses are, however, not required here. Each matrix fully describes the two-port, and the elements of one matrix may be recalculated from the elements of every other matrix. In the general case, the four matrix elements are independent of each other. However, for passive RLCT-two-ports (consisting exclusively of resistors, inductors, capacitors and transformers), only three degrees of freedom remain [e.g. 7], so that two matrix elements are dependent on each other in a simple fashion. Such two-port networks are called transmission-symmetric or **reciprocal**. Using so-called technical♣ reference arrows [7], the following holds for reciprocal two-ports:

$$Z_{12} = -Z_{21}, \quad Y_{12} = -Y_{21}, \quad H_{12} = H_{21}, \quad G_{12} = G_{21}$$

Conditions of reciprocity

Plugging these conditions into the two-port equations and generating, with $U = 0$ or $I = 0$, a rogue starting state, simple relationships between the operation in the forward and the reverse directions result.

| | |
|---|---|
| $U_1/I_2 = Z_{12}$ $(I_1 = 0)$    $Z_{21} = U_2/I_1$ $(I_2 = 0)$ | $I_1/U_2 = Y_{12}$ $(U_1 = 0)$    $Y_{21} = I_2/U_1$ $(U_2 = 0)$ |
| $U_1/U_2 = H_{12}$ $(I_1 = 0)$    $H_{21} = I_2/I_1$ $(U_2 = 0)$ | $I_1/I_2 = G_{12}$ $(U_1 = 0)$    $G_{21} = U_2/U_1$ $(I_2 = 0)$ |

As an **example**: The retroaction $H_{12} = U_1 / U_2$ identified for primary open circuit ($I_1 = 0$) corresponds to the current amplification $H_{21} = I_2 / I_1$ established for secondary short circuit ($U_2 = 0$).

♣ Besides the technical reference-arrow system, applying the **symmetric reference-arrow system** would be just as justifiable; in that case, the direction of the output current would be reversed, and in the conditions of reciprocity, all signs would be reversed. The symmetric reference-arrow system is not used in this book.

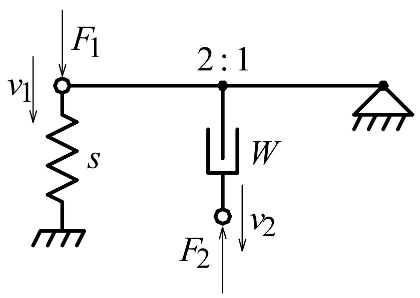
 

In more general terms: knowledge of the transmission behavior in one direction (e.g. from pole-pair 1 to pole-pair 2, or from port 1 to port 2) enables us to determine the transmission behavior into the reverse direction (from port 2 to port 1).

A pickup is not a purely electrical two-port, but this does not stand in the way of a consideration of reciprocity: given the transducer equations $\underline{F} = \alpha \cdot \underline{U}$ and $\underline{I} = \alpha \cdot \underline{v}$, the mechanical signal quantities may be recalculated into the electrical signal quantities, and using $\underline{Z}_{el}$ = $\underline{Z}_{mech}$ / $\alpha^2$, the system quantities may be recalculated [3], such that the electromechanical transducer two-port is changed into a purely electrical reciprocal two-port. As a rogue connector-loading, $v = 0$ and $F = 0$ could theoretically be defined, with only $F = 0$ being practically significant; the total immobility is not obtainable precisely enough.

Of the four conditions of reciprocity, the equality of the G-parameters is best suitable to calculate the piezo pickup. Defining the connector pair designated with 1 as electrical side, and the pair designated with 2 as the mechanical side, the boundary conditions are: from $U_1 = 0$, $F = 0$ results (no external force, bridge pieces without load), and $I_2 = 0$ implies an electrical open circuit (the usual piezo-connection). Using the reciprocity-conditions, the force→voltage-transmission we are looking for can be deduced from the voltage→velocity-transmission (measureable via Laser-vibrometer).

As an **example** for the reciprocity conditions, let us examine an ideal lever to which a spring and a dampening resistance are mounted (**Fig. 6.10**). The reference system used here is an option – other reference systems are possible, as well.




$s = \underline{F}/\underline{x} = j\omega\underline{F}/\underline{v}$ ideal spring (Hooke)

$W = \underline{F}/\underline{v}$ ideal friction (Stokes)

small deflections ≈ linear operation.

**Fig. 6.10:** Mechanical 2-port purely as example; no direct bearing to the piezo pickup.

The boundary conditions relating to $G_{12} = G_{21}$ are reformulated (using $\underline{F} = \alpha \cdot \underline{U}$ and $\underline{I} = \alpha \cdot \underline{v}$) into a first operational state with mechanical open-circuit ($F_1 = 0$), and into a second operational state with fixed output ($v_2 = 0$). This yields:

$$\frac{\underline{v}_2}{\underline{v}_1} = \frac{2s}{j\omega W} + \frac{1}{2} \quad \text{for } F_1 = 0 \qquad \text{and} \qquad \frac{\underline{F}_1}{\underline{F}_2} = \frac{2s}{j\omega W} + \frac{1}{2} \quad \text{for } v_2 = 0$$

In the 1st case the excitation happens at the right-hand connector (index 2), and the velocity-ratio is determined; in the 2nd case excitation occurs at the left-hand connector, and the force-ratio is determined. In both cases the same transmission function results – but only for the specified load conditions, and not for all load-types.

The system specified above can be reformulated as an equivalent electrical system using the algorithms of electromechanical analogies [e.g. 3]. From the possible choices FI- and FU-analogies, we select the latter because it corresponds to the physical transducer principles present in a piezo transducer. For the network-structure, however, we need to consider that not an isomorphic but a **duality**-network results (the flow-quantity force is mapped to the potential-quantity voltage [3]). The FU-analogy converts a spring into a capacitance, a friction-resistance into an electrical resistor, and a lever into an ideal (inductance-free) transformer that can also transmit DC. For a lever, the ratio of the lever arms (and thus $v_1/v_2$) is stated; a transformer, however, has – corresponding to the duality – a *voltage* ratio, and thus the transformation ratio is inverted (**Fig. 6.11**).

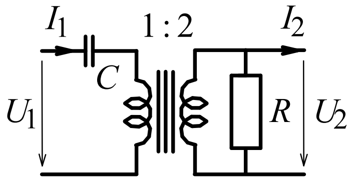



**Fig. 6.11:** Electrical equivalent circuit diagram derived via FU-analogy from Fig. 6.10. Equal velocity results in equal current, i.e. series connection; equal force results in equal voltage, i.e. parallel connection (dual network)

For the two load conditions $U_1 = 0$ and $I_2 = 0$, a transmission function can be defined for the above; the same result is obtained with the conversion $C = \alpha^2/s$ and $R = W/\alpha^2$ :

$$\frac{\underline{I}_2}{\underline{I}_1} = \frac{2}{j\omega RC} + \frac{1}{2} \quad \text{for } U_1 = 0 \qquad \text{and} \qquad \frac{\underline{U}_1}{\underline{U}_2} = \frac{2}{j\omega RC} + \frac{1}{2} \quad \text{for } I_2 = 0 \qquad \Diamond$$

Contrary to a purely mechanical system (Fig. 6.10) or a purely electrical system (Fig. 6.11), the **piezo pickup** represents an electromechanical system. For such a system, the relations of reciprocity hold, too. In the ideal transducer (**Fig. 6.12**), we immediately note that the transmission factors are the same: for primary open-circuit ($F_1 = 0$), we have $I_2 / v_1 = \alpha$, and for secondary open-circuit, the transducer factor $F_1 / U_2 = \alpha$ results. This sameness even holds for every load condition in the ideal transducer – and therefore naturally also for the boundary conditions of the reciprocity-relations.

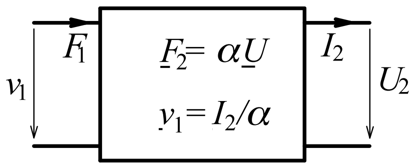



**Fig. 6.12:** Ideal piezo transducer

As opposed to an ideal piezo transducer, the real piezo transducer includes mechanical and electrical components that are to be connected as a mechanical two-port on the left side and as an electrical two-port on the right side. The two-ports are of reciprocal character as far as they merely include masses, springs, friction-resistances and levers, or inductances, capacitances, resistors and transformers, respectively, and the overall system is then reciprocal, as well. A corresponding two-port ladder-network is shown in **Fig. 6.13** – it may, of course, again be consolidated into a single two-port. Given the boundary conditions mentioned above, we obtain, for $F_1 / U_2$ and $I_2 / v_1$, the same coefficient of proportionality. However, the latter does not correspond anymore to the transducer constant $\alpha$, but depends (in a possibly complicated fashion) on the frequency. This dependency can be measured relatively easily in the $I_2 \rightarrow v_1$-operation, and may be carried over to the $F_1 \rightarrow U_2$-operation (that is more difficult to measure).

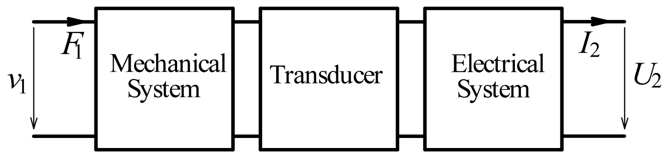



**Fig. 6.13:** Real piezo transducer

For the $I_2 \rightarrow v_1$-operation, a generator with a low-impedance AC-output is connected to the electrical input of the pickup. Since the pickup represents (in good approximation) a purely capacitive electric load independent of its mechanical loading, it is easy to make the connection from the electrical voltage $\underline{U}_2$ to the current $\underline{I}_2$ ($\underline{I}_2 = j\omega C\underline{U}_2$). This current→velocity transmission-coefficient $T_{vI}$ corresponds to the force→voltage transmission-coefficient $T_{UF}$ for the electrical open-circuit:

$$\frac{\underline{U}_2}{\underline{F}_1} = \frac{\underline{v}_1}{\underline{I}_2} = \frac{\underline{v}_1}{j\omega C \cdot \underline{U}_2} = \frac{\underline{x}_1}{C \cdot \underline{U}_2} \qquad T_{UF} \text{ for } I_2 = 0, \quad T_{vI} \text{ for } F_1 = 0$$

The oscillation-velocity $v_1$ can be determined e.g. with a Laser-vibrometer; due to the small values to be measured, suitable averaging is mandatory.

## 6.5 Operation as an actor

Piezo-electric materials convert in both directions: mechanical quantities into electrical ones (operation as sensor), and electrical quantities into mechanical ones (operation as actor). As an electric AC-voltage is connected to the electrical connectors of the pickup, the bridge piece vibrates up and down … a bit. A very small bit, actually: merely a few nanometers. We could not find out at which voltage the crystal was going to receive irreversible damage, and therefore the following measurements were carried out with a RMS-voltage of 10 V – no recognizable damage occurred there. During the measurement, the Ovation guitar was placed in its case, and the vibration velocity was measured using a **laser-vibrometer** (Polytec). Based on the equivalent circuit diagram shown in Fig. 6.8, we would expect, for a mass-free bridge piece (idealization), a frequency-independent *displacement*, if a frequency independent voltage is imprinted. However, the vibrometer – based on the Doppler effect – measures the vibration velocity as its source-quantity, and therefore the measurement grows more difficult with decreasing frequency. Nevertheless, using sufficiently narrow-band filters makes coherent results possible also in the low frequency domain (**Fig. 6.14**). Both the actor- and the sensor-measurements show, as a 1$^{st}$-order approximation, a frequency-independent transmission factor, although there are smallish frequency peaks – these are mainly caused by the guitar and not by the measuring process.

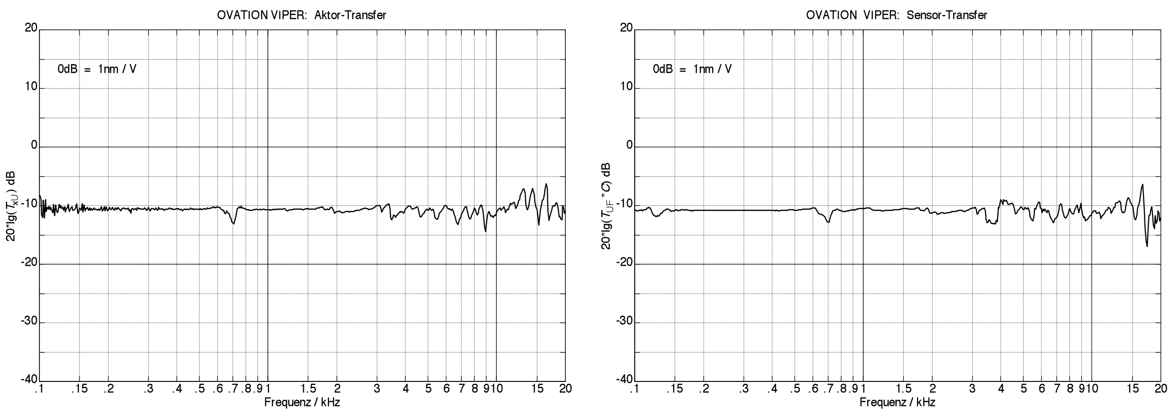



**Fig. 6.14:** Transmission factor $G_{xU}$ as measured with the laser-vibrometer (left). For comparison, the corresponding sensor-measurement is shown on the right: $T_{xU} = C \cdot T_{UF}$. The correspondence is impressive. "Frequenz" = frequency; "Aktor" = actor.