Supplement to Chapter 6: piezo-electric equations of state

The description of piezo-electric energy conversion uses generally accepted **formula symbols** that, however, often have a different implication in other frameworks. The symbol *S*, for example, elsewhere often stands for a surface area; for the piezo crystal, however, it designates the relative deformation (strain). The letter *E* finds much use as indicator for field strength, but also represents energy, or the modulus of elasticity. The letter *d* may serve to designate a diameter, or a thickness – or the piezo-modulus. The following table lists the formula symbols required to describe piezo-electric transducers. In order to avoid ambiguities, the definition of some of these quantities is limited to the present special chapter.

Symbol	Unit	Designation		Symbol	Unit	Designation
A	m^2	(Surface-) Area		L	H	Inductance
C	F	Capacitance		m	kg	Mass
d	m / V	Piezo-modulus		Q	As	Electrical charge
D	As / m^2	Dielectric displacement		R	Ω	Electrical resistance
e	N / Vm	Piezo-force constant		s^E, s^D	m^2 / N	Elasticity-coefficient♣
E^E, E^D	N / m^2	Modulus of elasticity♣		S	1	Relative deformation
E	V / m	Electric fields-strength		t	s	Time
F	N	Force		T	N / m^2	Mechanical stress
g	Vm / N	Piezo-voltage constant		U	V	Electrical voltage
h	V/m	-		W	Ws	Energy
k	1	Coupling factor		α	N / V	Transducer constant
l	m	Length		ε^S , ε^T	F / m	Dielectric constant♣

In the indexing, the three **coordinates in space** (*x*, *y*, *z*) are replaced by the numbers 1, 2, 3 – in agreement with common convention, the direction perpendicular to the vibrating surface of a thickness-mode oscillating block is indexed with the index 3 (**Fig. 6A.1**). The thickness-mode oscillator is the transducer type most often found in pickups; in the following only this type will be discussed. Designated here with *l*, the thickness typically amounts 0.2 … 1 mm in many cases, while the surface *A* will be about 0,1 … 1 cm^2. Except for the thickness-mode oscillator, there are also flexural oscillators, planary-mode oscillators, shear-mode oscillators, and more types. The thickness-mode oscillator is sometimes also called longitudinal oscillator, or thickness-oscillator with longitudinal effect – the terms are not consistent.

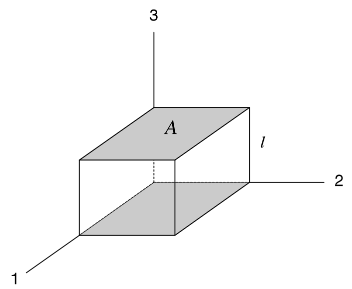


Fig. 6A.1: Piezo-crystal with directional definitions. Top and bottom are metalized and have the surface *A* each; between the surfaces the electrical field strength is formed. The height (thickness) of the crystal is *l*. For the thickness-mode oscillator shown here, the movement (and force) occurs in the vertical direction indexed with 3. The electrical fields run correspondingly in parallel.

♣ The superscript letters designate special load conditions – this will be elaborated on later.

The piezo-crystal shown in Fig. 6A.1 is subjected to the force F_3 in the vertical direction, and a vertical mechanical stress $T_3 = F_3 / A$ results. This, however, is the **first particular case** that needs to be specially designated: *only* one single mechanical loading is to be present – without any electrical loading. The electrical voltage therefore needs to be set to zero via a short across the electrodes (the metalized connecting surfaces). Because, due to this short, the electrical field strength is zero, this special condition is designated with a superscript E – which must not be seen as a mathematical power. Electrically shorted, the crystal reacts as if the piezo-effect did not exist, i.e. for static loading the crystal merely reacts like a spring:

$$S_3 = \Delta l / l = T_3 / E_{33}^E = T_3 \cdot s_{33}^E ; \quad T_3 = F_3 / A \qquad \text{Hooke's law}$$

In this equation, S_3 designates the relative change in length often termed ε in mechanics. However, since ε is also the symbol for the dielectric constant (that is required here, as well), we term – in electro-mechanics – the relative length-change with S. Specifically, we term it with S_3 because it is directed vertically in the direction indexed with 3. The mechanical stress occurring in the vertical direction is designated with T_3 (normally in mechanics the symbol would be σ). E_{33}^E stands for the modulus of elasticity without piezo-effect, i.e. for $E = 0$. The double-indexing (33) is required in the framework of general considerations, because electrical vectors (first number) and mechanical vectors (second number) can occur in different directions. For the present considerations, however, we have a limitation to the vertical direction (see figure), so that the indexing could be dispensed with. Even though, we will keep it in order to maintain conformity to literature. The superscript E is not a mathematical power but a reference to the boundary condition of the electrical field strength: $E = 0$. The formula symbol E (for the modulus of elasticity) can easily be mixed up with an E designating the field strength; we therefore normally use, rather than the modulus of elasticity, its inverse s with the same indexing. Thus, s_{33}^E does not represent a stiffness here, but it indicates the elasticity-coefficient in the direction 3 for the field strength set to zero. While this nomenclature requires getting used to, it is also found in datasheets (e.g. Siemens piezo-ceramics).

To summarize: if we disable the piezo-effect by electrical shorting, we obtain – for static mechanical loading – a **crystal acting like a spring** in the direction 3, having an elasticity coefficient s_{33}^E, the (purely mechanical) behavior of which is described by Hooke's law.

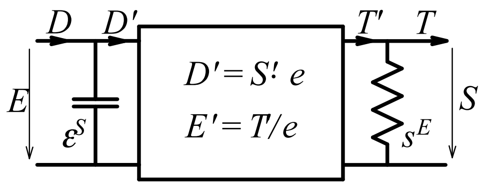


Fig. 6A.2: The piezo-crystal under static loading. Contrary to Chapter 6.1, the area-specific and length-specific quantities are given here. Quadripole arrows in the technical direction. $E = E'$, $S = S'$.

We must now account for the fact that the electrical side will of course not always be shorted. For the **second particular case** with purely electrical loading that is looked into now, any contribution from the mechanical side is prevented by the condition $S = 0$. This mechanical short circuit, also called "firmly-braked condition", necessitates complete rigidity on the mechanical side: the relative change in length (change in thickness) needs to be zero. Conceptually, this can be achieved with an infinitely stiff crystal.

It is customary to indicate this lack of mechanical deformation (S = 0) in the dielectric constant ε by a superscript S:

$$D_3 = Q/A = \varepsilon_{33}^S \cdot E_3 = \varepsilon_{33}^S \cdot U/l = C_K \cdot U/A \qquad \text{Capacitor-equation}$$

In summary: if we disable the piezo-effect via a mechanical short, we obtain – for static loading in the direction 3 – a **capacitor** with the dielectric constant ε_{33}^S; the (purely electric) behavior of this capacitor is described by the capacitor equation.

In the third and last step, we drop the particular conditions of the purely mechanical or purely electrical loading; we now arrive at the general operation by superposition of the two particular cases. The *ideal* piezo-electrical transducer-process (as shown in **Fig. 6A.2** by the rectangle) connects electrical and mechanical quantities:

$$D' = S' \cdot e_{33} \qquad E' = T' / e_{33} \qquad \text{Differential transducer-equations}$$

Using the reference arrows defined in Fig 6A.2, the two node-conditions read:

$$T' = T + T_s \qquad D' = D - D_\varepsilon \qquad \text{Node-equations}$$

The two-pole equations of the storage-elements connect flow- and potential-quantities:

$$T_s = S / s_{33}^E \qquad D_\varepsilon = E \cdot \varepsilon_{33}^S \qquad \text{Two-pole-equations}$$

From this, we can deduce the system of equations for the general operational case:

$$D = D_\varepsilon + D' = E \cdot \varepsilon_{33}^S + S \cdot e_{33}$$
$$T = T' - T_s = E \cdot e_{33} - S / s_{33}^E$$

$$\begin{pmatrix} D \\ T \end{pmatrix} = \begin{pmatrix} \varepsilon_{33}^S & e_{33} \\ e_{33} & -1/s_{33}^E \end{pmatrix} \cdot \begin{pmatrix} E \\ S \end{pmatrix}$$

The matrix shown on the right maps the two potential-quantities (E, S) onto two flux-quantities (D, T). It may be interpreted as conductivity-matrix, and transformed to the chain-matrix; at the same time new piezo-coefficients (d, e, g, h) are defined, as well:

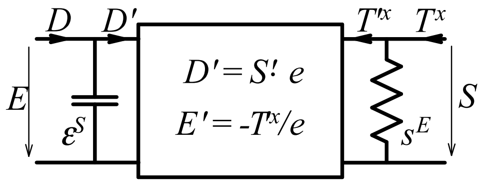
$$E = S/(e_{33} \cdot s_{33}^E) + T/e_{33}$$
$$D = S \cdot \left(e_{33} + \varepsilon_{33}^S/(e_{33} \cdot s_{33}^E)\right) + T \cdot \varepsilon_{33}^S / e_{33}$$

$$\begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} 1/d & 1/e \\ 1/g & 1/h \end{pmatrix} \cdot \begin{pmatrix} S \\ T \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} S \\ T \end{pmatrix}$$

$$d = e_{33} \cdot s_{33}^E; \qquad e = e_{33}; \qquad 1/g = e_{33} + \varepsilon_{33}^S/(e_{33} \cdot s_{33}^E); \qquad h = e_{33}/\varepsilon_{33}^S;$$

The chain-matrix **A** connects both quadripole input-quantities (E, D) to both output quantities (S, T). Its determinant $[\det(\mathbf{A}) = 1/dh - 1/ge = -1]$ is negative because we have a gyratorical mapping here: the ideal transducer maps the potential quantities (E', S') onto the flux-quantities (D', T') [3]. The correspondingly specified signs are in agreement with the four-pole-theory, but in contradiction to the piezo-parameters normally stated in datasheets – these parameters are based on old IEEE-recommendations.

Using the **definition of the algebraic sign** as it is customary in datasheets, the determinant of the chain matrix will be +1, and not -1 (as it would be specific for a gyrator). Now, there are several possibilities to invert the signs in a series connection of three quadripoles. The conversion from the technical direction of the arrows to the symmetrical direction is easy to interpret. This simply corresponds to reversal of the direction of the forces – a direction that requires special attention anyway. Reversing the direction of the forces also reverses the reference direction of the perpendicular stress T specific to the surface-area – the reversed variant of which is termed T^x in the following (Fig. 6A.3).



$$\begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} 1/d & -1/e \\ 1/g & -1/h \end{pmatrix} \cdot \begin{pmatrix} S \\ T^x \end{pmatrix}$$

Fig. 6A.3: The piezo crystal under static stress. Flow-arrow at the output reversed relative to Fig. 6A.2).

The quadripole equivalent-circuit as shown in Fig. 6A.3 uses the symmetric arrow-direction (as it is common in quadripole theory for the **X**-, **Y**-, **H**- and **G**-matrix) for the definition of the chain-matrix. This is not the usual approach but it is a way to arrive at compatibility with the datasheets. The individual mappings are as follows:

$$\begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon_{33}^S & 1 \end{pmatrix} \cdot \begin{pmatrix} E' \\ D' \end{pmatrix} \qquad \begin{pmatrix} E' \\ D' \end{pmatrix} = \begin{pmatrix} 0 & -1/e \\ e & 0 \end{pmatrix} \cdot \begin{pmatrix} S' \\ T'^x \end{pmatrix} \qquad \begin{pmatrix} S' \\ T'^x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/s_{33}^E & 1 \end{pmatrix} \cdot \begin{pmatrix} S \\ T^x \end{pmatrix}$$

Consolidating the three matrices into an overall matrix via a multiplication, we get:

$$\begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} \dfrac{1}{e_{33}s_{33}^E} & \dfrac{-1}{e_{33}} \\ \dfrac{\varepsilon_{33}^S}{e_{33}s_{33}^E} + e_{33} & \dfrac{-\varepsilon_{33}^S}{e_{33}} \end{pmatrix} \cdot \begin{pmatrix} S \\ T^x \end{pmatrix} = \begin{pmatrix} 1/d & -1/e \\ 1/g & -1/h \end{pmatrix} \cdot \begin{pmatrix} S \\ T^x \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} D \\ T^x \end{pmatrix} = \begin{pmatrix} \varepsilon_{33}^S & e_{33} \\ -e_{33} & 1/s_{33}^E \end{pmatrix} \cdot \begin{pmatrix} E \\ S \end{pmatrix}$$

In this representation, the determinant of the chain-matrix is +1, as is customary in the datasheets. The piezo-parameters can be converted as follows:

$$d = e \cdot s^E = g \cdot \varepsilon^T; \qquad e = d/s^E = h \cdot \varepsilon^S; \qquad g = h \cdot s^D = d/\varepsilon^T; \qquad h = g \cdot c^D = e/\varepsilon^S;$$

$$s^E - s^D = d^2/\varepsilon^T = g^2 \cdot \varepsilon^T = d \cdot g; \qquad \varepsilon^T - \varepsilon^S = d^2/s^E = e^2 \cdot s^E = d \cdot e$$

The superscript letters in these formulas refer to setting the respective quantity to zero, i.e. for example ε^T = dielectric constant for zero-ed (mechanical) perpendicular stress T. However, this always refers to external quantities: in Fig. 6A.3, this would be $T^x = 0$ and not $T'^x = 0$, and $D = 0$, not $D' = 0$. For the other signal quantities, this distinction is not necessary because of $E = E'$ and $S = S'$. The subscript indices need to be included if the orientation in space is to be specified: for example d_{33} (thickness-mode oscillator), d_{15} (thickness-shear-mode oscillator), d_{25} (surface-shear-mode oscillator).

Besides the description of the system with differential quantities (referring to length and area), there is also an integral (macroscopic, global) representation in which *U*, *I*, *v*, *F* are used rather than *E*, *D*, *S*, *T*. The electrical field strength *E* is the length-specific electrical voltage *U*, the mechanical stress *T* is the area-specific force *F*:

$$E = U/l \qquad T = F/A \qquad I = A \cdot j\omega D \qquad v = l \cdot j\omega S$$

For the other two quantities, a time derivative (or an integration, respectively) is required – in the spectral representation, this corresponds to a multiplication with (or, respectively, a division by) *jω*. All signals are complex, as is always the case in general signal theory; the under-strike often is dispensed with:

$$I = dQ/dt \;\Leftrightarrow\; \underline{I} = A \cdot j\omega \underline{D} \qquad v = d\xi/dt = l \cdot dS/dt \;\Leftrightarrow\; \underline{v} = l \cdot j\omega \underline{S}$$

The microscopic description using differential quantities looks at the scenario of static stress (*f* = 0). Here, a velocity – the vibration- (particle-) velocity – would be of little help, though, and therefore its integral referring to the length (the relative deformation *S*) is used. Correspondingly, the current strength *I* (which is zero in the case of a static load) is replaced by its integral referring to the area, i.e. the charge density (displacement density) *D*. This static load condition is, however, less relevant for the practical deployment: the electrical resistances cannot be increased indefinitely, and therefore there is always some current flowing that leads to recharging processes. For this reason, calculations in practice mostly use *U*, *I*, *v*, *F*. Applying integral notation, the equations for the ideal piezo transducer read:

$$F' = U' \cdot \alpha \qquad v' = I'/\alpha \qquad \alpha = eA/l \qquad \text{Integral notation}$$

The algebraic signs are oriented towards Fig. 6A.2, and therefore do not correspond to datasheet-conventions. The apostrophes express that the ideal transducer-effect is referred to – without any contributions by mechanical or electrical two-poles. A block diagram is shown in **Fig. 6A.4** – contrary to Fig 6A.2 it now includes the integral quantities. The two-pole parameters change correspondingly:

$$C^v = \frac{A \cdot \varepsilon^S}{l} \qquad n^U = \frac{l}{A \cdot s^E} \qquad C = Q/U \qquad n = \xi/F$$

For the “firmly-braked”, mechanically fixed case ($S = 0$, or $v = 0$), the dielectric constant ε^S becomes the capacitance C^v, for the electrical short-circuit case ($E = 0$, or $U = 0$) the elasticity constant s^E becomes the spring-compliance n^U.

Cave!: in this chapter, s^E does not designate the stiffness (= force / deflection) but the elasticity coefficient (= 1 / modulus of elasticity)!

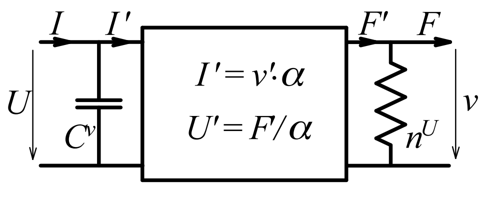


Fig. 6A.4: Block-diagram of the piezo transducer. The capacitance shown in the figure is the crystal-capacitance from Fig. 6.1 B ($C^v = C_K$), the shown compliance is reciprocal to the stiffness of the crystal: $n^U = 1 / s_K$.