A. Appendix: Vibrations, Waves, and the Cryo-Schlock

A.1 Vibrations and waves

A **vibration** (or oscillation) is a process in which the vibration quantity (e.g. displacement, force, current) is non-monotonic and has at least two extreme values (DIN 1311). In mechanics, oscillations occur when masses move around their resting position. Masses are elements of mechanical systems, as are springs and dampers. If a system contains no dampers, it is lossless or undamped, otherwise it is lossy or damped. A mass can store kinetic energy; a spring can store potential energy. A damper is not one of the elements that store energy; the energy that is fed to it is irreversibly converted into heat (caloric energy). The number of all independent energy-storage elements determines the **order** n of the system.

A mass that can swing up and down hanging from a spring is a second order system. In order to fully describe the state of the vibration of the system at *one single* point in time, n state variables (field quantities) are necessary. In the case of the spring-mass system, this can be, for example, displacement and velocity. In systems-theory, the state variables are also called **signal quantities**; they are to be distinguished from **system quantitys** (mass, stiffness, impedance...). Physical laws of force/displacement, and structural laws (topology, Kirchhoff's laws for nodes and meshes) determine the interactions between the system elements, and lead to a differential equation (**DE**), the solution of which yields the oscillation equation. An nth order system is defined by a DE of n-th order. For mechanical systems with discrete (locally concentrated) elements, the independent variable is usually the time t; the signal variables are represented as being dependent on t in a time function.

Vibrations that occur (exclusively) under the influence of an external excitation source are called **forced** vibrations. If the source oscillates mono-frequently with f_0, and the system is linear, then all state variables are sinusoidal and have the frequency f_0. The term *sinusoidal* allows for any phase angle, including a cosine curve. With external excitation, the descriptive DE is inhomogeneous, i.e. it contains a so called 'constant term' (which need not to be constant). Without any excitation at a given time (= homogeneous DE), the system is either constantly at rest, or it 'responds' to previous excitations (it continues to oscillate: it 'rings'). The ability of a vibrating system to ring is due to the ability to store energy in its mass(es) and spring(s). The **ringing** occurs at an Eigen-frequency ("natural frequency" being an equivalent term) of the system, and here the terminologies of mechanics and systems-theory differ: in mechanics, the damped oscillation is described as a product of exponential function and sine (or cosine) to which one frequency is assigned. Systems-theory recognizes a period in the decay process, but no periodicity (!), and assigns an infinitely wide continuous spectrum via the Fourier integral. This is because spectral analysis is decomposition into summands, not into factors [see literature on systems theory, e.g., 6].

If the properties of an oscillatory system are continuous functions of the place, then the system is called a **continuum**. Its state variables are functions of time and place (space); the composite of individual particle vibrations is called a **wave**. The dependence on two variables leads to partial DE's, which in turn can be homogeneous (for the free wave) or inhomogeneous (for the forced wave).

A.1.1 Forced vibration

In the forced (externally excited, source-excited) oscillation, energy is supplied to the oscillation system from the outside. For example, the displacement $\xi(t)$ of a mass suspended from a spring can be impressed sinusoidally with the frequency f_0. This simultaneously defines the velocity, acceleration, and other temporal differentials and integrals. The force acting between mass and spring can be calculated both from Newton's law of inertia, and from Hooke's suspension law.

The vibrations are not easy to grasp if the displacement is not impressed on the mass end of the spring but on the opposite (upper) end of the spring. The mass now performs oscillations with f_0, but with initially unknown amplitude and phase. In terms of systems theory, there is a transmission – a mapping – from an input variable (displacement of the spring) to an output variable (displacement of the mass). In the time domain, the problem is solved by the DE or by convolution with the impulse response, and in the frequency domain by the transfer function [6]. Considerations along the lines of the analogies between electrical and mechanical systems allow electrical signal and systems theory to be applied to mechanical systems as well [3]. External excitation does not mean that *only* the forced vibration must exist. Despite externally forced excitement, a free oscillation can exist at the same time. This free oscillation represents a response to previous excitations. In the case of the linear system (linear DE), both oscillations overlap without influencing each other.

A.1.2 Free vibration

After external excitation, a free oscillation can form at a natural frequency (Eigen-frequency) of the system. First-order systems do not generate self-oscillation♣ but exponential transient processes exclusively. Second-order systems show exactly one natural frequency; systems of higher order usually have several natural frequencies. In the case of the linear system (linear DE), all possibly existing natural oscillations (equivalent term: “Eigen-oscillation”) are superimposed undisturbed. Each Eigen-oscillation is characterized by its natural frequency (system quantity), its initial amplitude and its phase (signal quantity), and its damping (system quantity). In the undamped system, the damping is zero (infinite Q-factor): the vibration does not decay. In the damped system, each natural vibration decays exponentially. Special cases arise in case of coincident natural frequencies (multiple poles).

EXAMPLE: A second order system has the natural frequency f_E. A source with a frequency $f_1 \neq f_E$ delivers an excitation signal, with the source being switched on for the duration $-T \le t < 0$, and being switched off at $t = 0$. How does the system react? It should be noted that the switched sine (burst) contains not only the frequency f_1, but rather all frequencies (→ Fourier-Integral). During $-T$ ... 0, a forced oscillation is generated, afterwards we have a free oscillation. The frequency f_E and damping of the latter are determined by the system characteristics, the oscillation amplitude and phase result from the values of two state variables at the time $t = 0$. If between T_2 ... T_3 >0, the source is switched on again, but now operates with the frequency f_2, the second forced oscillation is superimposed on the first decaying oscillation. After T_3, two free oscillations (with the same frequency f_E) are superimposed. This example assumes that the switching does not change the system characteristics, i.e. that in particular the source impedance and the structure remain the same. ◊

♣ DIN 1311 calls creep processes "vibration in a broader sense". More precise definitions → systems theory.

The solutions of the differential equations set out in the preceding chapters do not yet give any special indications of the actual string vibrations. For their detailed description of the latter, string geometry and string excitation must also be known. String geometry includes string length and bearing data; with excitation, a forced wave is generated; without (or after) excitation, we get a free wave. With guitars, a forced wave occurs only in the short moment when the moving plectrum makes contact with the string (attack). The following decay process is a free wave - at least as long as the string does not hit the frets or is plucked again.

The differential equation (DE) required to describe the string vibration is initially a partial one because the dependent variable (e.g., ξ) depends on location z and time t. The DE is linear, because the variables occur only in the first power. Without consideration of the bending stiffness, a 2nd-order DE results, with consideration of the bending stiffness we get a 4th-order DE. If the system quantities (e.g., stiffness, density, geometry) maintain their values over time – as it is assumed here - the DE contains constant coefficients. Of particular importance are **linear DE's with constant coefficients** (linear/time-invariant systems or **LTI systems**), because for them the **principle of superposition** holds. According to this principle, any complicated space- or time-function may be seen as the sum of individual partial oscillations, with these partial oscillations not influencing each other. Free and forced vibrations or waves may exist at the same time – they then overlap in their effect.

In the general case, forced and free oscillations (or waves) superimpose on the string. Mathematics express this as follows: *The general solution of an inhomogeneous linear DE is given as the sum (superposition) of a particular solution of the inhomogeneous DE and the general solution of the homogeneous DE.*

A.1.3 Forced Wave

In the forced wave, energy is supplied to a continuum (e.g. a string) from the outside. Since the string vibration depends on space and time, complicated excitations may be formulated. The following simplification is important in practice: the time-dependent excitation happens at *one place*. For plucking/picking with a pointed plectrum, this approximation represents a first step, with Chapter 1.5 showing detailed results.

Further simplification is required in the definition of the excitation quantity. Rigid strings are described by force, moment, displacement and angular velocity. Each of these quantities can act (singly or in combination) as an excitation variable. The description may be simplified if the excitation quantity can be defined as the direct string input quantity. We would have indirect excitation if, for example, the defined movement of a point (finger) acts on the string via a spring (plectrum). In the case of direct excitation, the variable acting directly on the string (e.g. displacement) is defined via its temporal progress.

Systems theory assigns a source impedance (internal impedance) of value zero or infinity to the direct excitation; the indirect excitation receives an intermediate value. Direct excitation (**impression** of a signal quantity) is easier to describe. Particularly descriptive is the impressing of the transverse displacement (at the constant location z_0).

 Translation provided by Volker Eichhorst (with Tilmann Zwicker)

Suppose, for example, that the transverse displacement $\xi(z_0,t)$, as function of t, is impressed at a given location z_0. Given this, several questions can be formulated:

a) According to which function does the displacement $\xi(z,t)$ develop at another location z?
This is a standard problem of systems theory, solved via impulse response and convolution (see below).
b) Which function holds for the other signal variables (F, M, etc.)?
At one and the same location, power-generating quantities (*F*, *v* or *M*, *w*) are linked by means of the string-impedances; the cross-linking is defined by coupling terms. The mapping to another location is solved as in a).
c) Which oscillation occurs after the excitation has ended?
The result is a free wave, with its starting conditions defined by the final conditions of the preceding excitation.

With respect to a): The Bernoullian approach takes as a solution the harmonic exponential, which yields sinusoidal space- and time-functions. Since in the previous calculations the string could be modeled as being lossless, the amplitude of a mono-frequency oscillation remains the same during the propagation, and only the phase changes. Thus, at the location z, again only a sinusoidal oscillation of the same frequency and the same amplitude can arise, the phase of which is rotated relative to the excitation phase. Since the propagation velocity for the homogeneous string (with location-independent string parameters) is independent of z, the phase rotation is proportional to the distance traveled. Moreover, in the case of a string without bending stiffness, the propagation velocity is frequency-independent (linear-phase system), and thus the phase rotation is proportional to the frequency and the distance. In the rigid string, dispersion occurs: the phase grows over-proportionately with increasing frequency.

Arbitrary non-sinusoidal signals need to be decomposed into their sinusoidal components (Fourier analysis), which are then mapped individually from z_0 to z; subsequently the transformed components are reassembled (Fourier synthesis). Alternatively, the entire mapping can also be performed in one step with the **convolution integral**: for this purpose, the excitation time signal is to be convoluted with the impulse response of the string. For the string without bending stiffness, the impulse response is a time-shifted Dirac impulse (dispersion-free delay-time system, delay line); for the stiff string, it is an all-pass function (Chapter 2, or [6]). Reflections can be modeled as system responses to additional mirror sources, and can then be superimposed.

As Fig. 1.19 shows, the pick/string contact can be a few milliseconds long for normal plucking/picking. This is the length of time during which a forced vibration exists. The excitation spreads over the entire string during this time (Fig. 1.10), so that in a strict sense we may no longer speak of an impulse- or step-stimulus.

In most cases, the sound generated by the guitar-amp/loudspeaker can feed back to the guitar. If high gain is employed, a significant excitation of the guitar string happens via the airborne sound generated by the speaker. This can lead to self-excitation (howling feedback). The overall system is active and nonlinear in this case; the oscillation can no longer be termed free but rather self-exciting.

A.1.4 Free wave

Free waves are solutions of the homogeneous differential equation (Chapter 2). Looking at the matter with a cursory glance, we might suspect that free waves can only exist if there are no external forces. Indeed, external forces cause forced waves to arise – however, free waves can always exist *in addition*. Since these waves contain kinetic and potential energy, they cannot emerge out of nowhere, though.

The partial wave-DE has two variables: time and space. If both the periodicities with respect to space and time are imposed from the outside, we speak of a space-time forced wave – abbreviated: **forced wave**. If the external excitation occurs only within a limited space, so-called **spatially free waves** build up outside of this space. If the external excitation is limited in time, then a space-time free wave is created – in short a **free wave**.

In the previous chapters the differential equations were set up without external forces (homogeneous DE), their solutions describe free oscillations. As an example, we will examine a transverse wave (Fig. A3.1):

$$\Psi \cdot \frac{\partial^2 \underline{\xi}}{\partial z^2} = m' \cdot \frac{\partial^2 \underline{\xi}}{\partial t^2} \qquad \underline{\xi} = \hat{\xi} \cdot e^{j\varphi} \cdot e^{j(\omega t - kz)} \qquad \text{DE and solution}$$

The solution is a harmonic oscillation with amplitude $\hat{\xi}$, initial phase φ, and angular frequency ω. To prove that the solution fits the DE, we differentiate $\underline{\xi}$ twice with respect to space and time, and insert these derivatives into the DE:

$$\Psi \cdot \left(-k^2 \underline{\xi}\right) = m' \cdot \left(-\omega^2 \underline{\xi}\right) \qquad \} \qquad \Psi \cdot k^2 = m' \cdot \omega^2 \qquad \text{Characteristic equation}$$

The equation on the left may be truncated for each $\underline{\xi}$ by $\underline{\xi}$ ($\underline{\xi} \equiv 0$ is the trivial case and not of interest). The characteristic equation yields the wave number k as a function of the angular frequency, and of the string parameters. If k is inserted into the solution, the latter still contains the three parameters amplitude $\hat{\xi}$, initial phase φ, and angular frequency ω. This result indicates that transversal waves of *any* amplitude, initial phase and angular frequency can propagate on the (infinitely long) string. They can – but do not have to! The actually existing vibrations depend on the preceding excitation – the latter defines the initial conditions, and thus the above-mentioned parameters.

A.1.5 Standing waves

A standing wave results from two waves of equal frequency but opposed propagation direction being superimposed in one area at the same time. The complex notation of the wave equation gives us:

$$\underline{\xi}(z,t) = \hat{\underline{\xi}} \cdot \left(e^{j(\omega t + kz)} + e^{j(\omega t - kz)}\right) = \hat{\underline{\xi}} \cdot \hat{e}^{j\omega t} \cdot 2\cos(kz); \qquad e^{j\varphi} + e^{-j\varphi} = 2\cos\varphi$$

Here, the cosine can be interpreted as a place-dependent amplitude term: at the zeros of the cosine function, nodes of the standing wave arise; at their maxima, antinodes arise. The distance between two adjacent nodes or antinodes, respectively, is one half (!) wavelength.

If the amplitudes of the waves propagating in opposite directions are exactly equal, the result is a pure standing wave; otherwise there is a combination of standing waves and progressive waves (the latter also called travelling waves). A progressive wave carries energy in the direction of its propagation, while a standing wave does not.

There are forced standing waves and free standing waves, depending on whether or not an excitation source is present. With the guitar, the plucking/picking takes only a few milliseconds; after that, the progressive waves reflected again and again at the bearings generate a free standing wave, the amplitude of which slowly fades away. The Eigen-frequencies (natural frequencies) of this standing wave are calculated from the cosine function given above; for the lowest Eigen-frequency, the nodal distance is precisely the string length. **Fig. A.1.1** below shows for comparison a progressive transverse wave and a standing transverse wave.

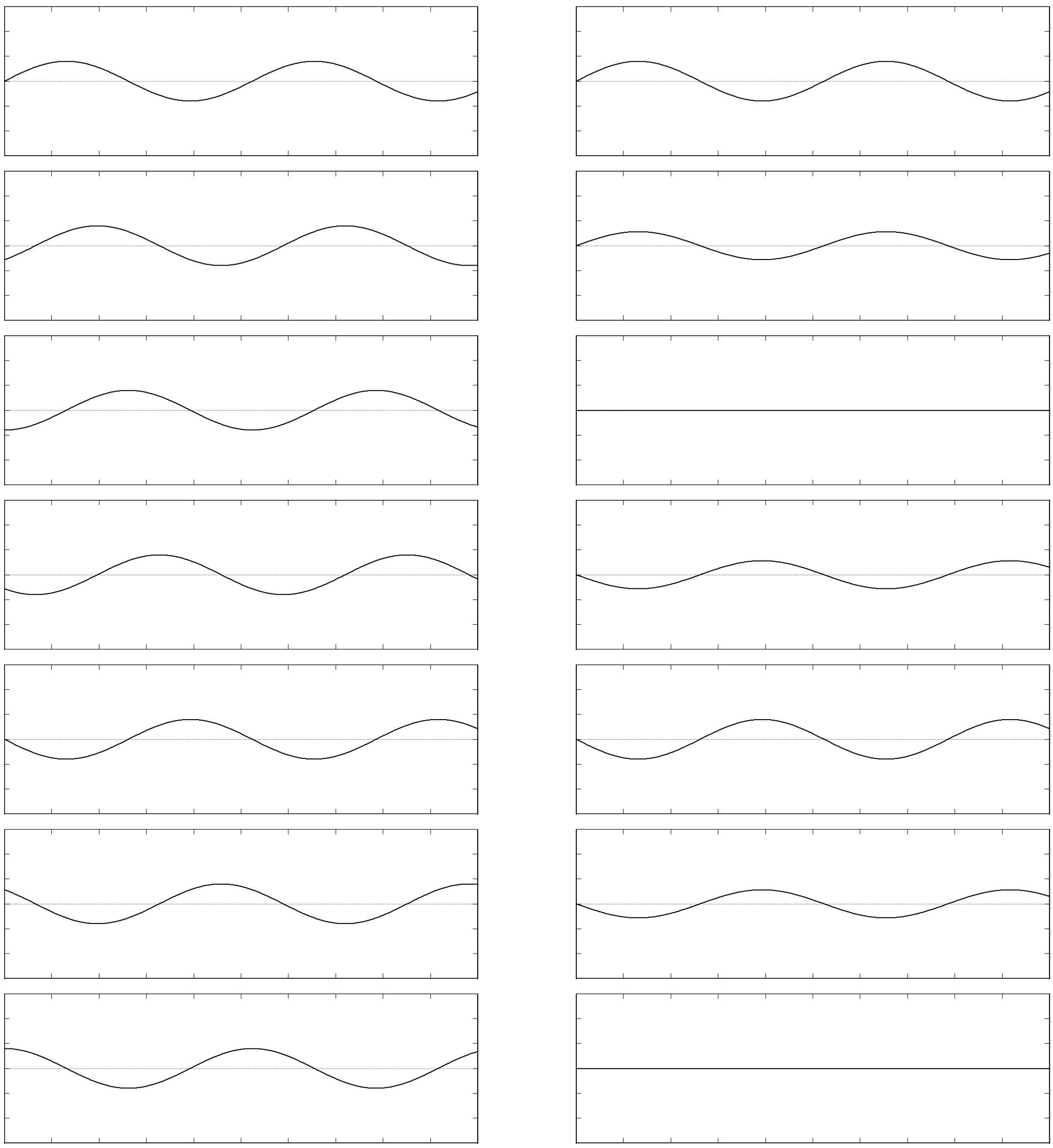

Fig. A.1.1: Progressive transverse wave (left), standing transverse wave (right); phase increment = π/ 4. See also: https://gitec-forum.de/wp/collection-of-the-animations/ or https://www.gitec-forum-eng.de/knowledge-base-2/collection-of-animations/.

