A.2 Longitudinal waves

In a longitudinal wave, oscillation direction and propagation direction coincide. (The transverse wave that is more important for the guitar is discussed below in A.3). For the calculation, we divide the transmission medium into thin disks via flat, transverse, equidistant separation surfaces. Given the wave propagation, these discs will change both their position in the propagation direction, and their thickness. The separation surfaces are perpendicular to the propagation direction, forming planes of equal normal stress, and equal displacement, respectively. The change in thickness is connected to the engaging force via Hooke's Law, which in turn is linked to the moving mass of the respective disk via Newton's law of inertia.

A.2.1 Pure longitudinal waves

The signal quantities (field quantities) are force F and longitudinal velocity v, as well as displacement and acceleration derived from these quantities. The system quantities are the material data s_L and ρ. The **longitudinal stiffness** s_L characterizes material deformations in the case of large transverse dimensions, i.e. with inhibited transverse contraction. With the guitar string, transverse dimensions are very small so that this load case does not occur. In the string, tension in the longitudinal direction rather leads to a length-wise extension of the string while reducing the diameter. Therefore, in addition to longitudinal oscillations, coupled thickness-oscillations also occur. The combination of both vibrations is called **dilatational wave**, contraction wave, or quasi-longitudinal wave.

A.2.2 Dilatational waves in strings

Dilatational waves (quasi-longitudinal waves) occur in transmission media that feature small transverse dimensions with respect to the wavelength, e.g. in plates, rods, or instrument strings. The *primary* forces and movements are parallel to the longitudinal axis of the string. However, *secondary effects* occur in the transverse direction perpendicular to the string axis: elongation in the longitudinal direction reduces the string diameter, compression increases it. While the percentile **changes of the transverse dimensions** are very small, they are still essential. In the purely longitudinal wave, the transverse dimensions remain constant while longitudinal forces act; this is only possible because lateral forces act at the same time (three-axis stress state). In the case of dilatational waves, only longitudinal forces (or longitudinal stresses) occur, and it is precisely for this state that the **elastic modulus *E*** was defined as a constant of proportionality. Relative change in length $\Delta z/z$, and the longitudinal stress σ_z (= longitudinal force / cross-sectional area) are proportional:

$$\varepsilon_z = \frac{\Delta z}{z} = \frac{\sigma_z}{E}; \qquad \sigma_x = \sigma_y = 0$$

E = Modulus of elasticity = Young's modulus

The change in diameter caused as a secondary effect in the x- and y-directions depends, via the relative lateral contraction μ (Poisson's ratio), on the longitudinal strain:

$$\varepsilon_x = \varepsilon_y = -\mu\varepsilon_z$$

Lateral contraction

The minus-sign is required because longitudinal increase in the dimensions results in a transversal decrease. The dimensionless Poisson's ratio μ is material-dependent – for steel it amounts to approximately 0.3.

The oscillation DE of the (lossless) dilatational wave can be established from Hooke's law and Newton's law:

$$E\frac{\partial^2 F_z}{\partial z^2} = \rho\frac{\partial^2 F_z}{\partial t^2}\,; \qquad E\frac{\partial^2 v_z}{\partial z^2} = \rho\frac{\partial^2 v_z}{\partial t^2}$$ Differential equation

This **differential equation** is of the same type for the longitudinal force F_z, and for the longitudinal velocity v_z or for its integral (displacement), or its differential (acceleration). For the solution of the DE's, the separation approach according to DANIEL BERNOULLI is suitable. In it, a time-dependent factor and a place-dependent factor are separated (using the example of the longitudinal force F_z below):

$$\underline{F}_z = \hat{F}_z \cdot e^{j\varphi} \cdot e^{j\omega t} \cdot e^{-jkz} = \underline{\hat{F}}_z \cdot e^{j(\omega t - kz)}$$ Proposed solution

Herein, F_z is the time- and place-dependent longitudinal force of the wave – it must not be confused with the tensioning force Ψ in the string. As usual in signal theory, F_z is formulated as a harmonic exponential, i.e. as a circulating pointer (phasor). Its projection onto the real axis (the real part of the complex quantity) corresponds to the actual force; the imaginary part (complementing the quantity to be of a complex magnitude) does not appear in practice. The complex representation nevertheless is not more of an effort, but rather makes for easier and shorter handling e.g. in integration / differentiation.

$\underline{\hat{F}}_z$ is the complex amplitude that contains the initial phase angle φ (at $t = 0$ and $z = 0$). The **angular frequency ω** is connected to the **time period T** via 2π, just as the **wave number k** is connected to the spatial periodicity (**wavelength λ**) via 2π. Both quantities are related via the **phase velocity c_P** (= propagation velocity):

$$k = 2\pi/\lambda \qquad \omega = 2\pi/T \qquad \omega/k = c_P = \sqrt{E/\rho}$$ Wave quantities

For the temporal partial differential, the location z is a constant – for the spatial partial differential the time t is a constant. For a fixed time t (flash-recording), the local force is of sinusoidal shape, as is the temporal force for a fixed location (force sensor). The term *sinusoidal* allows for any initial phase; the specific value is determined by the excitation signal. As long as the system is considered to be **linear** and **time-invariant** (LTI-system), any signal can be synthesized by superposition. This solution approach is therefore valid not only for sinusoidal vibrations, but for all waveforms. **Fig. A.2.1** shows a snapshot of a sinusoidal (mono-frequency) dilatational wave, in **Fig. A.2.2** different phase positions are shown for this purpose.

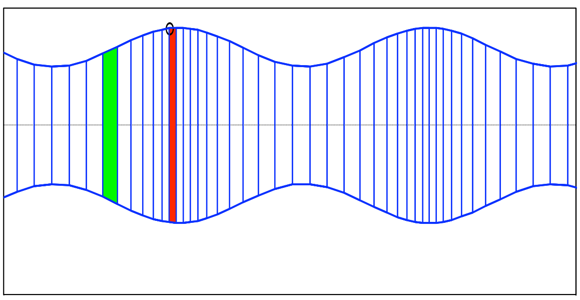

Fig. A2.1: Mono-frequency dilatational wave

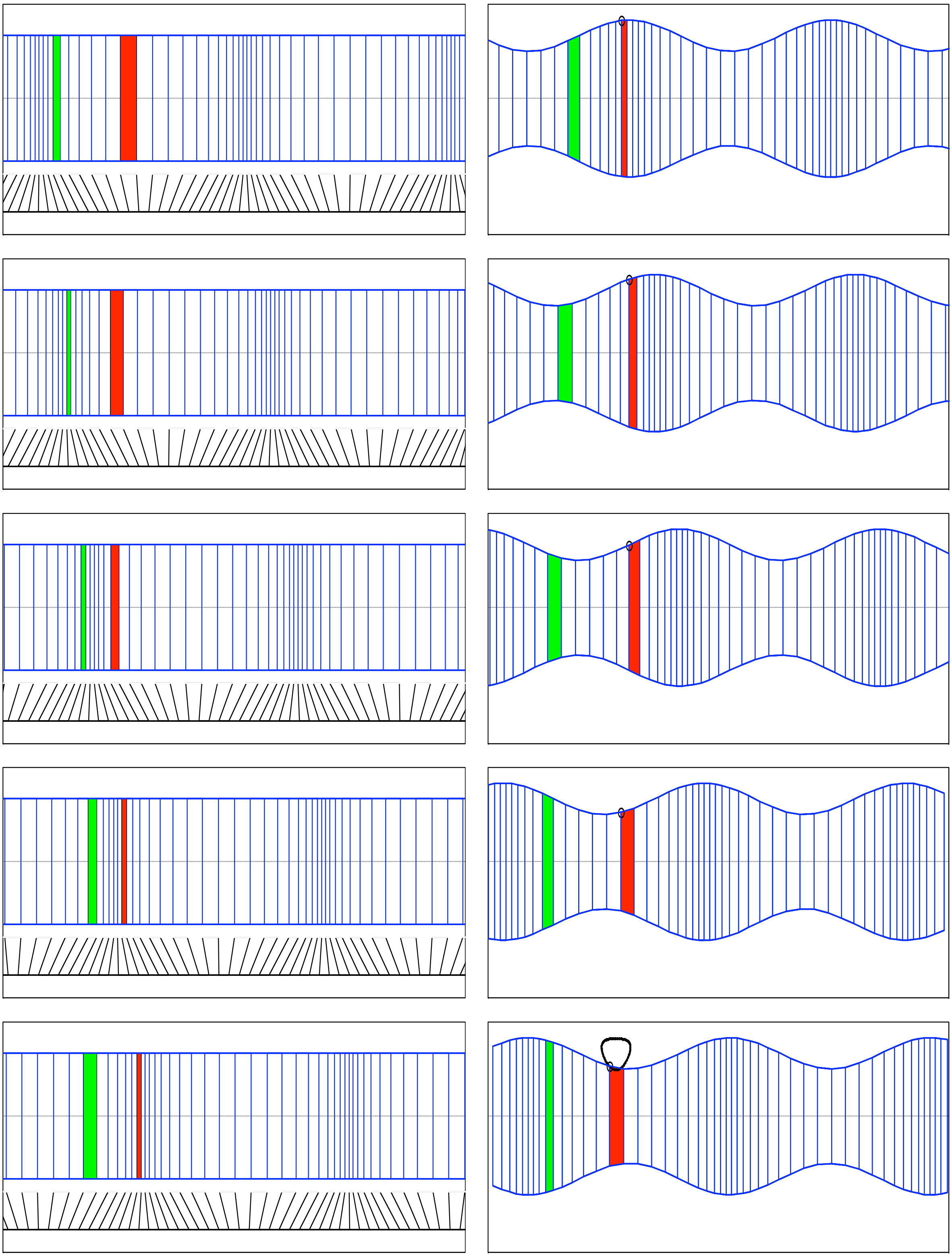

Fig. A.2.2: Progressive longitudinal wave (left), progressive dilatational wave (right). The phase increment is π/4. The densifications or dilutions run through the image from left to right, the partial volumes (in two cases colored) swing around their rest position. In the longitudinal wave, the displacement is indicated by oblique lines below the wave. The transverse constrictions of the dilatational wave are greatly exaggerated. See also: https://gitec-forum.de/wp/collection-of-the-animations/ or https://www.gitec-forum-eng.de/knowledge-base-2/collection-of-animations/.

Translation provided by Volker Eichhorst (with Tilmann Zwicker)