A.4 Bending waves (flexural waves)

The longitudinal and transverse waves discussed in the preceding chapters can be described by 2nd-order differential equations. However, more detailed investigations on vibrating strings show that in addition to the translational variables force and velocity, the rotational variables **bending moment** and **angular velocity** should also be taken into account. From this, a 4th-order differential equation results, which can now also describe **dispersive** (frequency-dependent) wave propagation.

A.4.1 Tension-free beam, pure bending wave

In the case of the pure transverse wave (A.3.1), the shear stiffness (described by the material quantity **shear modulus** G) counteracts the engaging shear forces. However, practical tests show that the effective stiffness is curvature-dependent, and this behavior is better described by a stress-loaded particle (of the medium) with nonparallel interfacing surfaces (Fig. A4.1). Since the effective transverse string dimensions (0.2-0.5 mm) are small compared to the wavelength (1-130 cm), shear deformations and rotational moments of inertia may be neglected (Euler-Bernoulli theory of the bending rod). The **bending stiffness** B results as the medium-characterizing quantity, being dependent on **Young's modulus** E, and on the axial geometrical moment (of inertia) I, the latter determined by of the **diameter** D of the cylindrical string:

$$B = E \cdot I = E \cdot D^4 \pi / 64 \qquad \text{Flexural stiffness}$$

For the solid string, D is the outer diameter; for the wound string D is, as a first approximation, the core diameter. For more detailed descriptions, the stiffness of the wrapping will also have to be taken into account to some degree, essentially depending on the tension under which the wrapping was applied.

To put together the differential equation, we divide the string into differentially small slices of the width dz. The circular separation surfaces are, in the rest state, perpendicular to the longitudinal axis (*z-axis*, **Fig. A.4.1**). Given excitation, they can change their position and direction, but they remain always flat and always perpendicular to the local (curved) axis of the string. The center of each separation surface can move in the direction perpendicular to z-axis ξ; in addition rotation in the image plane is permitted. The lateral displacement is denoted $\xi\,(z,t)$, the rotation $\beta(z,t)$. The mass dm of the discs enclosed between two adjacent separating surfaces always remains the same.

The direction β of each interface vector is at the same time the direction of the local string axis – the latter corresponding to the local derivative (slope) of the displacement ξ:

$$\beta = \frac{\partial \xi}{\partial z} \qquad v = \frac{\partial \xi}{\partial t} \quad w = \frac{\partial \beta}{\partial t} = \frac{\partial^2 \xi}{\partial t\, \partial z} = \frac{\partial v}{\partial z} \qquad \text{Motion quantities}$$

Here β is the rotation angle, $v = v_\xi$ is the (particle) velocity in the ξ-direction, and w is the angular velocity of the rotation. The angular velocity w must not be confused with the angular frequency ω! w is amplitude-dependent while ω is not.

The place dependence of the angle of rotation β causes a deformation of the disc-shaped slice of the string. From the science of strength of materials, we know that in the case of a straight bend, the curvature $\partial^2\xi/\partial z^2$ is linked to the moment M via the bending stiffness♣ B:

$$M = -B \cdot \frac{\partial^2 \xi}{\partial z^2}$$ Straight bend

The minus sign corresponds to a convention chosen in [11], textbooks in mechanics tend to write a plus sign here. If the sign definition is adhered to in the same way for all calculations, both considerations are equivalent.

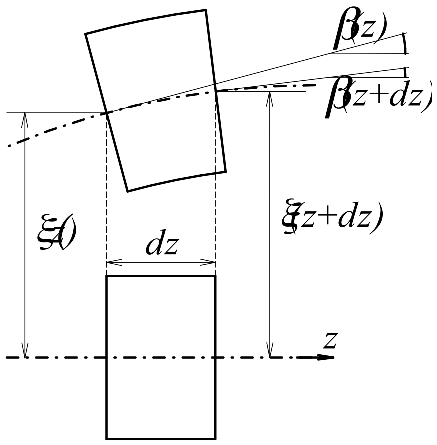
The relationship between the lateral force $F = F_\xi$ and the moment M is directly derived from its fundamental definition: a force F generates the moment $dM = Fdz$ with respect to an orthogonal axis spaced at a distance dz. In differential notation, with the sign convention according to [11]:

$$F = -\frac{\partial M}{\partial z}$$ Transverse force, moment

The axiom of inertia provides the relationship between transverse force, transverse acceleration and mass of the disc. Local differentiation with respect to z yields:

$$-\frac{\partial F}{\partial z} = m' \frac{\partial^2 \xi}{\partial t^2} \qquad m' = \rho \cdot D^2\pi/4 = \rho \cdot S$$ Inertia axiom

For solid strings, ρ is the material density, D is the diameter, S is the cross-sectional area. For wound strings, the outer diameter can be used for D, but we must then use an average density $\overline{\rho}$ for ρ, taking into account the proportion of air in the winding and, if necessary, the difference in density between the core and the winding (Chapter 1.2).



$$\beta(z) = \frac{\partial \xi}{\partial z}; \quad \frac{\partial \beta}{\partial z} = \frac{\partial^2 \xi}{\partial z^2}$$

$$\beta(z+dz) = \beta(z) + \frac{\partial \beta}{\partial z} dz$$

$$\xi(z+dz) = \xi(z) + \frac{\partial \xi}{\partial z} dz$$

Fig. A.4.1: Curved piece of string, deflected in the direction of ξ.

♣ The symbol M is also used for the scale length of the string, but not in this chapter.

If the two equations above for the straight bend and the transverse force are inserted into the inertia axiom, the differential equation for the pure bending wave results:

$$-B\frac{\partial^4 F}{\partial z^4} = m'\frac{\partial^2 F}{\partial t^2}$$ Differential equation

Instead of force F we could also use (on both sides of the equation) the moment M, the angle β, the angular velocity w, the angular acceleration $\dot{w}$, the lateral displacement ξ, the lateral velocity v, or the lateral acceleration $\dot{v}$. The solution is again based on the Bernoullian approach, assuming time- and location-functions of sinusoidal shape:

$$\underline{F} = \underline{\hat{F}} \cdot e^{j(\omega t - kz)} \qquad \frac{\partial^4 \underline{F}}{\partial z^4} = k^4 \cdot \underline{F} \qquad \frac{\partial^2 \underline{F}}{\partial t^2} = -\omega^2 \cdot \underline{F}$$ Solution

Substituting the two derivatives in the differential equation, we obtain for each F:

$$k^4 = m'\omega^2 / B$$ Characteristic equation

This 4th-order equation has 4 solutions. The positive real **wave number** k_1 describes a sinusoidal bending wave propagating in the positive z-direction, the negative real wave number k_2 describes a sinusoidal bending wave proceeding in the negative z-direction.

$$k_{1,2} = \pm\sqrt{\omega} \cdot \sqrt[4]{m'/B} \qquad k_{1,2} = \pm\, \omega / c_P$$ Wave number

Solving the last equation for the **phase velocity** c_P yields:

$$c_P = \pm\sqrt{\omega} \cdot \sqrt[4]{B/m'}$$ Phase velocity

c_P indicates the propagation velocity of a certain wave phase, e.g., a wave crest (maximum). As we can see, c_P depends on the frequency. Higher-frequency signal components run faster than low-frequency ones (**dispersion**). If, however, not a particular phase is of interest but the envelope maximum of a poly-frequency signal, it is not the phase velocity c_P that is decisive, but the group **velocity** c_G. It is twice as large as the phase velocity in the pure bending wave.

The third and fourth solutions of the characteristic equation are imaginary. It is expedient to define a real **fringe field number** k' with $k'_{1,2} = k_{3,4}/j$. With this, indexing can be simplified, so that only k and k' occur. For the pure bending wave, we simply have $k = \pm k'$; in the case of the rigid, tensioned string, the differences are greater (Chapter A.4.2).

$$k_{3,4} = \pm j\sqrt{\omega} \cdot \sqrt[4]{m'/B} = j \cdot k_{1,2}; \qquad k = \pm\sqrt{\omega} \cdot \sqrt[4]{m'/B}\,;\; k' = \sqrt{\omega} \cdot \sqrt[4]{m'/B}$$

$$\underline{F} = \underline{\hat{F}} \cdot e^{j\omega t} \cdot e^{-jkz} = \text{Wave} \qquad \underline{F} = \underline{\hat{F}} \cdot e^{j\omega t} \cdot e^{-k'z} = \text{fringe field } (z \geq 0).$$

In practice, only the fringe-field solution with *a negative* exponent can be used; it describes (with increasing distance to the bearing) a locally exponentially decaying fringe field.

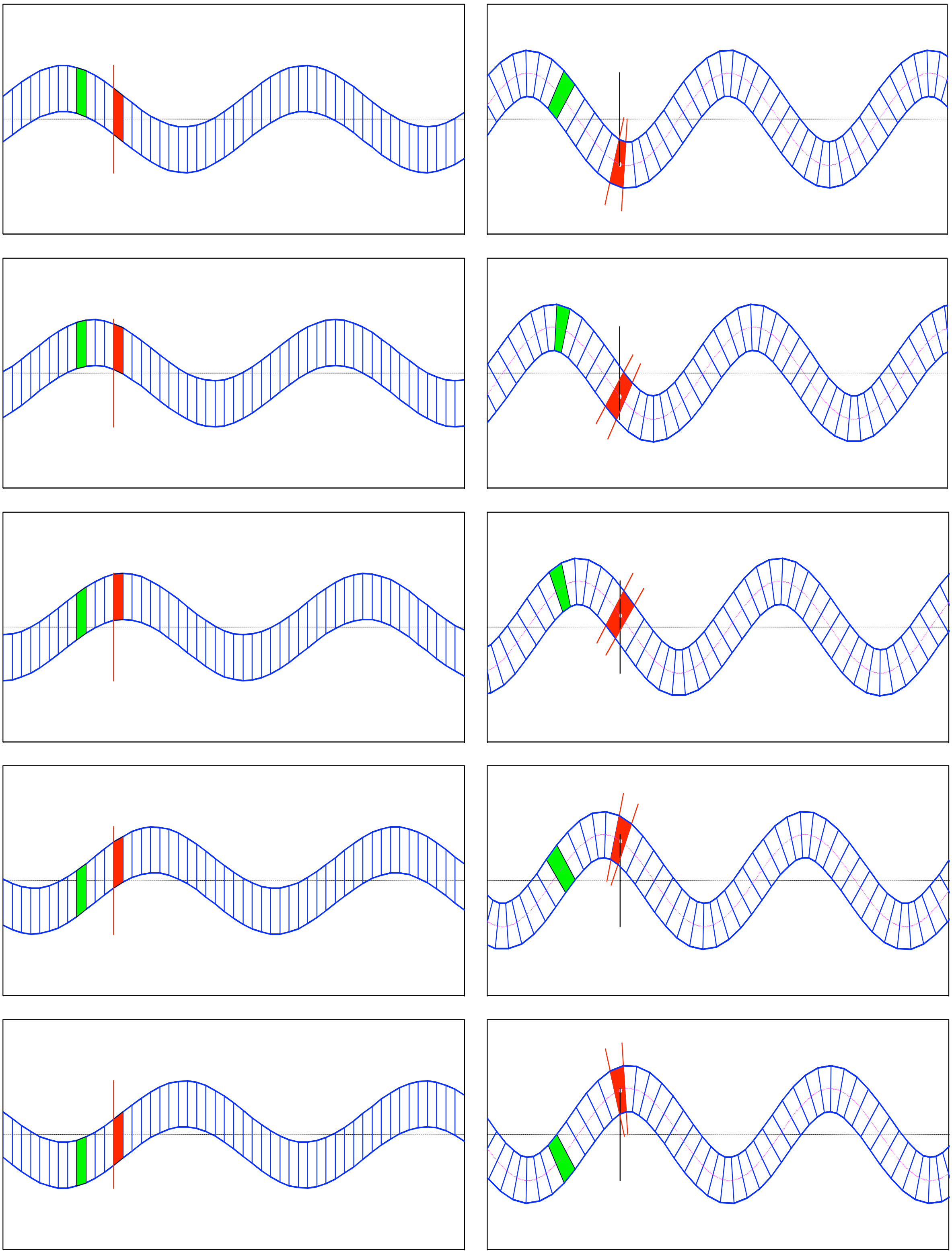


Fig. A.4.2: Progressive transverse wave (left), progressive bending wave (right). The phase increment is π/4 in each case. The transverse displacements run from left to right through the image, the partial volumes (colored in two cases) oscillate in transverse direction around their rest position. In the case of the transverse wave, the dividing lines (dividing surfaces) remain parallel; in the case of the bending wave, the angle between them varies. See also: https://gitec-forum.de/wp/collection-of-the-animations/ or https://www.gitec-forum-eng.de/knowledge-base-2/collection-of-animations/.

A.4.2 Rigid string with the tensioning force Ψ

The tensioning force generates the predominant part of the string stiffness (Chapter A.3.2). If this tensioning rigidity is supplemented by the bending stiffness (Chapter A.4.1), we already obtain a good approximation for the string vibration. At low frequencies, the tensioning rigidity very clearly dominates; at middle and high frequencies the additional bending stiffness becomes noticeable. At very high frequencies the model is no longer usable because the Euler-Bernoulli approximation no longer applies (→ Timoshenko [11]).

For the rigid, tensioned string, the stiffness formula presented in Chapter A.4.1 is to be supplemented to the tensioning stiffness:

$$F = -\frac{\partial M}{\partial z} - \Psi \frac{\partial \xi}{\partial z}$$ Transverse force

The differential equation thus receives an additional 2nd-order term:

$$\Psi \frac{\partial^2 F}{\partial z^2} - B \frac{\partial^4 F}{\partial z^4} = m' \frac{\partial^2 F}{\partial t^2}$$ Differential equation

As with the bending wave (Chapter A.4.1), M, β, w, $\dot{w}$, ξ, v, or $\dot{v}$ could also be used (instead of F). The Bernoullian approach provides the same solution as for the bending wave; however, the wave numbers k are different from the fringe field numbers k':

$$\underline{F} = \underline{\hat{F}} \cdot e^{j(\omega t - kz)} \qquad \omega^2 m' = \Psi k^2 + B k^4$$ Solution, characteristic equation

The equation – biquadratic in k – yields two real and two imaginary solutions. The two real solutions (k_1, k_2) describe waves propagating to the left or right, respectively; the two imaginary solutions (k_3, k_4) describe the (spatially!) exponentially increasing or decaying fringe fields (cf. Chapter A.4.1):

$$k_{1,2} = \pm\sqrt{\frac{1}{2B}\left(\sqrt{\Psi^2 + 4Bm'\omega^2} - \Psi\right)}\ ; \quad k_{3,4} = \pm j \cdot \sqrt{\frac{1}{2B}\left(\sqrt{\Psi^2 + 4Bm'\omega^2} + \Psi\right)}$$

As B approaches zero (flexible string, A.3.2), k converges to $\omega\sqrt{m'/\Psi}$ (limit value, from l'Hospital, or via a series expansion of the root); k' approaches infinity – this corresponds to a fringe field approaching zero. On the other hand, if Ψ converges to zero (bending beam, A.4.1), we obtain (same as with the pure bending wave):

$$k_{1,2} = \pm\sqrt{\omega} \cdot \sqrt[4]{m'/B} \qquad k_{3,4} = \pm j\sqrt{\omega} \cdot \sqrt[4]{m'/B} = j \cdot k_{1,2}$$ Pure flexural wave

With simplified indexing, the result for the **tensioned rigid string** is:

$$k = \pm\sqrt{\frac{1}{2B}\left(\sqrt{\Psi^2 + 4Bm'\omega^2} - \Psi\right)} \qquad k' = \sqrt{\frac{1}{2B}\left(\sqrt{\Psi^2 + 4Bm'\omega^2} + \Psi\right)}$$

A.4.3 Flexural Eigen-oscillations (natural oscillations)

The plucking of the string causes the progressive waves to propagate in both directions. A large number of resulting reflections overlap to form a standing wave. Since the string is a localized continuum, DIN 1311 nevertheless recommends that the corresponding term should not be called self-oscillation, but **Eigen-oscillation** or **natural oscillation**. Alrighty then… After plucking, each point of mass of the string can perform a free Eigen-oscillation with an Eigen-frequency of the string. The Eigen-frequencies (natural frequencies) are system quantities, and as such remain independent of the excitation. The oscillation amplitude, however, is a signal quantity with a value is determined by the excitation-signal and -location.

Each oscillation with a natural frequency is a mono-frequency process based on sinusoidal time functions. However, the DE of the flexural wave contains a 4^{th}-order place dependency, and therefore non-sinusoidal location functions can arise. Eigen-vibration modes and Eigen-frequencies result from four boundary conditions (two per bearing each). Amplitude and phase of the vibration result from the excitation signal.

At the string bearing (bridge, nut/fret), lateral force $\underline{F}$ and lateral velocity $\underline{v}$ are linked by the translation impedance $\underline{Z}_{FL}$. For example, the translation impedance caused by a mass m amounts to $\underline{Z}_{FL} = \underline{F}/\underline{v} = j\omega m$. Independent of this, the bearing for rotary motion has the rotational impedance $\underline{Z}_{ML}$. For example, an inertia moment Θ causes a rotational impedance $\underline{Z}_{ML} = \underline{M}/\underline{w} = j\omega\Theta$, linking the signal quantities of moment $\underline{M}$ and of angular velocity $\underline{w}$. In the general case, a coupling between translational and rotational motion is to be expected.

Simple **boundary conditions** arise when the bearing impedances converge to zero or to infinity. An infinite bearing mass leads to transverse-displacement, -velocity, and –acceleration jointly approaching zero. In the case when a translational impedance approaches zero, a transverse movement may, but there is no transverse force. Analogous relationships exist for the rotational movements. The combination of translational and rotational boundary conditions results in the following special cases:

- Free end: lateral force F and moment M are zero.
- Clamped end: transverse velocity v and angular velocity w are zero.
- Guided end: angular velocity w and shear force F are zero.
- Supported end: transverse velocity v and moment M are zero.

A closer analysis reveals that the bearing itself is a continuum capable of Eigen-oscillations; the bearing impedances are therefore not constants, but frequency-dependent. The different natural oscillations of a particular string can therefore encounter different (i.e. frequency-dependent) boundary conditions.

In the following, the flexural Eigen-vibration of a string clamped at both ends is examined. The tensioning force is initially assumed to be zero (**pure flexural** wave). The solution of the DE holds four components (in the range $z > 0$): a wave (jkz) running towards the bearing (assumed to be located at $z = 0$), another wave ($-jkz$) running away from the bearing, a fringe field ($-k'z$) decreasing with increasing x, and another fringe field ($k'z$) increasing with increasing x. The latter is meaningless in practice. The other three components of the particular solution derive their complex amplitude from excitation and bearing impedances.

From the DE, as well as from the associated characteristic equation and its four solutions $(\pm k, \pm k')$, the general solution results as a superposition:

$$\underline{\xi}(z,t) = \underline{\xi}_1 \cdot e^{+jkz} + \underline{\xi}_2 \cdot e^{-jkz} + \underline{\xi}_3 \cdot e^{-k'z} \qquad \text{General solution}$$

The place-dependent and time-dependent transverse displacement $\underline{\xi}$ depends on two waves and one fringe field. The time dependency results from the complex amplitudes $\underline{\xi}_i$, the place dependence results from the *e*-functions. We can regard the wave $\underline{\xi}_1$ running towards the bearing as an excitation that the other two components are dependent on:

$$\underline{\xi}_2 = \varsigma \cdot \underline{\xi}_1 \qquad \underline{\xi}_3 = \gamma \cdot \underline{\xi}_1 \qquad \underline{y}(x,t) = \underline{y}_1 \cdot \left(e^{+jkx} + \varsigma \cdot e^{-jkx} + \gamma \cdot e^{-k'x} \right)$$

With ς and γ, the last equation contains two parameters the value of which is determined by the two bearing impedances. To calculate, the DE of the transverse displacement needs to be converted onto the other state variables ($\partial / \partial t \rightarrow j\omega, \quad \partial / \partial z \rightarrow jk$):

$$\underline{v} = \frac{\partial \underline{y}}{\partial t} = j\omega \underline{y}_1 \cdot \left(e^{+jkx} + \varsigma \cdot e^{-jkx} + \gamma \cdot e^{-k'x} \right) \qquad \text{Velocity}$$

$$\underline{w} = \frac{\partial \underline{v}}{\partial x} = j\omega \underline{y}_1 \cdot \left(jk e^{+jkx} - jk\varsigma \cdot e^{-jkx} - k'\gamma \cdot e^{-k'x} \right) \qquad \text{Angular velocity}$$

$$\underline{F} = B \frac{\partial^3 \underline{y}}{\partial x^3} = \underline{y}_1 \cdot B \cdot \left[jk^3 \cdot \left(-e^{+jkx} + \varsigma \cdot e^{-jkx} \right) - k'^3 \gamma \cdot e^{-k'x} \right] \qquad \text{Force}$$

$$\underline{M} = -B \frac{\partial^2 \underline{y}}{\partial x^2} = \underline{y}_1 \cdot B \cdot \left[k^2 \cdot \left(e^{+jkx} + \varsigma \cdot e^{-jkx} \right) - k'^2 \gamma \cdot e^{-k'x} \right] \qquad \text{Moment}$$

The bearing conditions for the location $z = 0$ can now be used in these four equations. For example, a free bearing requires: $\underline{F}(0) \equiv 0, \ \underline{M}(0) \equiv 0$. The left side of the equation thus is zero; it can be reduced by $\underline{\xi}_1$, and the signal-<u>in</u>dependent system quantities ς and γ can now be determined. An analogous approach provides the reflection- and boundary-field-parameters for the other special cases:

Free end:	$\underline{F}(0) \equiv 0$	$\underline{M}(0) \equiv 0$	$\varsigma = -j, \quad \gamma = +1 - j$
Clamped end:	$\underline{v}(0) \equiv 0$	$\underline{w}(0) \equiv 0$	$\varsigma = -j, \quad \gamma = -1 + j$
Guided end:	$\underline{F}(0) \equiv 0$	$\underline{w}(0) \equiv 0$	$\varsigma = +1, \quad \gamma = 0$
Supported end:	$\underline{v}(0) \equiv 0$	$\underline{M}(0) \equiv 0$	$\varsigma = -1, \quad \gamma = 0$

Here, ς constitutes the **complex reflection factor**. For ς = 1, the reflection is in phase, for $\varsigma = -1$ it is in opposite phase; at $-j$ it is phase-shifted by -90°. γ is a **fringe field factor**. For the last two of the above wave terminations, no fringe field is generated.

The fringe field is strongest at $z = 0$, but decays exponentially with growing z. A **limit distance** can be defined as the distance from the fringe at which a drop to $1/e$ occurs; with the guitar string, this is only a few millimeters. The *amount* of the reflection factor ς is always 1 for the compiled special cases; this is to be expected because the reflection happens without damping.

At the **supported end**, transverse motion and moment are zero; rotational motion and lateral force may or may not be zero. Since there is no fringe field, the place-function is of sinusoidal shape. The oscillation looks like a standing transversal wave; the Eigen- Frequencies are, however, not an integer multiple of the fundamental frequency. **Fig. A.4.3** shows the displacement $\xi(z)$ for different Eigen-frequencies:

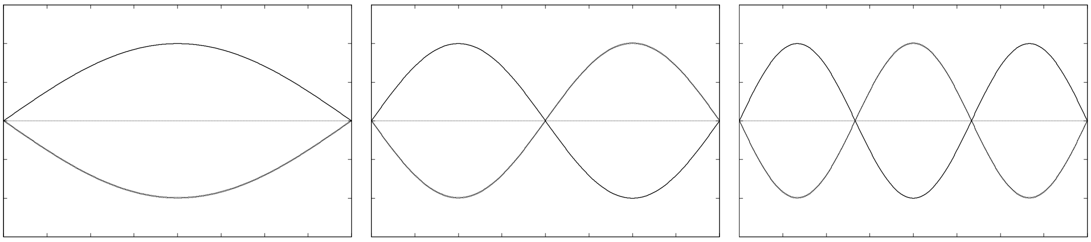


Fig. A.4.3: Standing flexural wave, supported ends, order $n = 1, 2, 3$.

At the **guided end**, the rotational movement and the lateral force are equal to zero, the lateral movement and the moment may or may not be zero. Since there is no fringe field, the place-function is of sinusoidal shape. The oscillation looks like a standing transversal wave, but the Eigen-frequencies are not an integer multiple to the fundamental frequency. **Fig. A.4.4** shows the displacement $\xi(z)$ for different Eigen-frequencies:

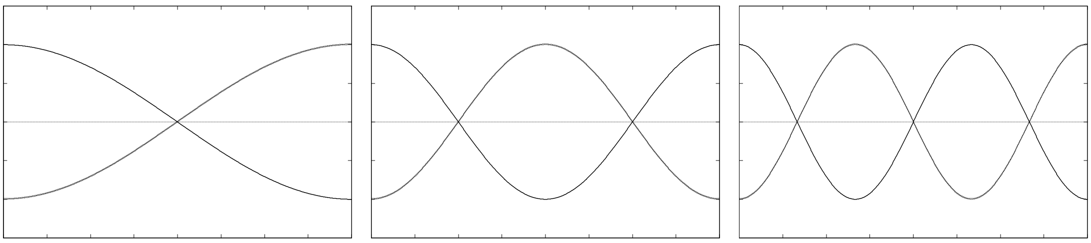


Fig. A.4.4: Standing flexural wave, guided ends, order $n = 1, 2, 3$.

At the **free end**, moment and lateral force are equal to zero, lateral movement and rotational movement may or may not be zero. Since there is a fringe field, the place-function is non-sinusoidal. **Fig. A.4.5** shows the displacement $\xi(z)$ for different Eigen-frequencies:

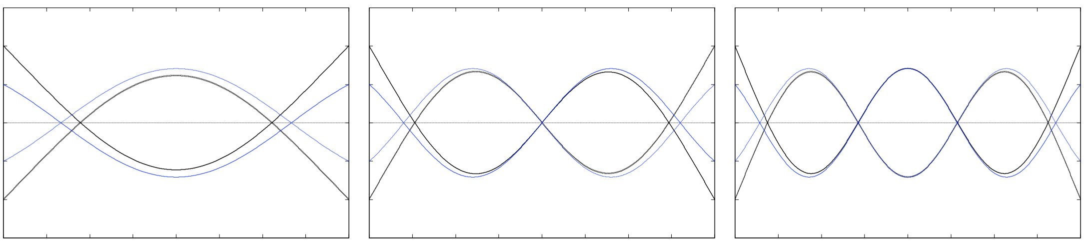


Fig. A.4.5: Standing flexural wave, free ends, order $n = 2, 3, 4$. The thin curves show the vibration without fringe field. In reality, the case $n = 1$ cannot occur as Eigen-oscillation.

At the **clamped end**, transverse motion and rotational motion are zero, moment and lateral force may or may not be zero. Since there is a fringe field, the place-function is non-sinusoidal. **Fig. A.4.6** shows $\xi(z)$ for different natural frequencies:

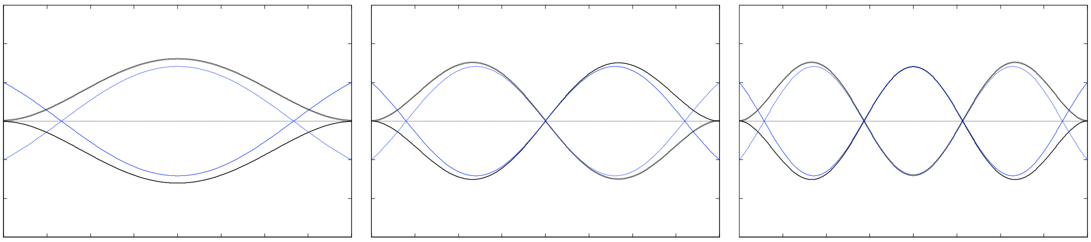


Fig. A.4.6: Standing flexural wave, clamped ends, order n = 1, 2, 3. The thin curves show the vibration without fringe field.

Due to the frequency-dependent propagation velocity, the Eigen-frequencies of the flexural vibration deviate considerably from those of the transversal oscillation. For the bar supported on both sides or clamped on both sides (i.e. where there are no fringe fields), we obtain [11]:

$$f_n = \frac{\pi}{2}\sqrt{\frac{B}{m'}}\frac{n^2}{L^2}$$ n-th order Eigen-frequency

For the bar free on both sides or clamped on both sides, the Eigen-frequencies are calculated approximately (with only one-sided consideration of each fringe field [11]) as follows:

$$f_n \approx \frac{\pi}{2}\sqrt{\frac{B}{m'}}\frac{(n-0{,}5)^2}{L^2}$$ n-th order Eigen-frequency

With guitar strings, the bending stiffness is to be considered mainly for large diameters and in the higher frequency range – and even there still only to a small extent. The predominant portion of the rigidity is generated by the clamping force; Figs. A.4.5 and A.4.6 correctly show the general influence of the fringe fields – but greatly exaggerated if this were a guitar-string scenario. Formally, the difference between the pure flexural wave and the **rigid string** with regard to the force is taken into account via the following:

$$\underline{F} = B \cdot \frac{\partial^3 \underline{\xi}}{\partial z^3}$$ Beam

$$\underline{F} = B \cdot \frac{\partial^3 \underline{\xi}}{\partial z^3} - \Psi \cdot \frac{\partial \underline{\xi}}{\partial z}$$ String

However, if the bearing conditions are not as ideal as in the special cases mentioned above, differences with regard to the flexible string may occur in the low-frequency range, as well (blocking mass, total passage, Chapter 2.5.2).

