A.5 Wave impedances (characteristic impedances)

For the progressive transversal wave, the **characteristic impedance** Z_W (equivalent term: wave impedance) connects the transverse force F to the transverse velocity v. For the **idealized** (rigidity-*free*) string, Z_W depends on the length-specific mass m' and the length-specific compliance n':

$$Z_W = F / v = \sqrt{m'/n'} = \sqrt{\Psi \cdot \rho \cdot D^2 \pi / 4} = \frac{\pi}{2} \rho \cdot M \cdot f_G \cdot D^2 = \Psi / c = m' \cdot c \qquad \text{Wave impedance}$$

In this case, we have the following correspondences: Ψ = clamping force, ρ = density, D = diameter, M = scale♣, f_G = fundamental frequency, c = phase velocity. Common Z_W-values are between 0.1 and 0.4 Ns/m for solid strings. For wound strings, the flexural stiffness may be ignored in the low frequencies-range; ρ may simply be replaced by $\overline{\rho}$. The associated wave impedances are then in the range between 0.3 and 1.2 Ns / m.

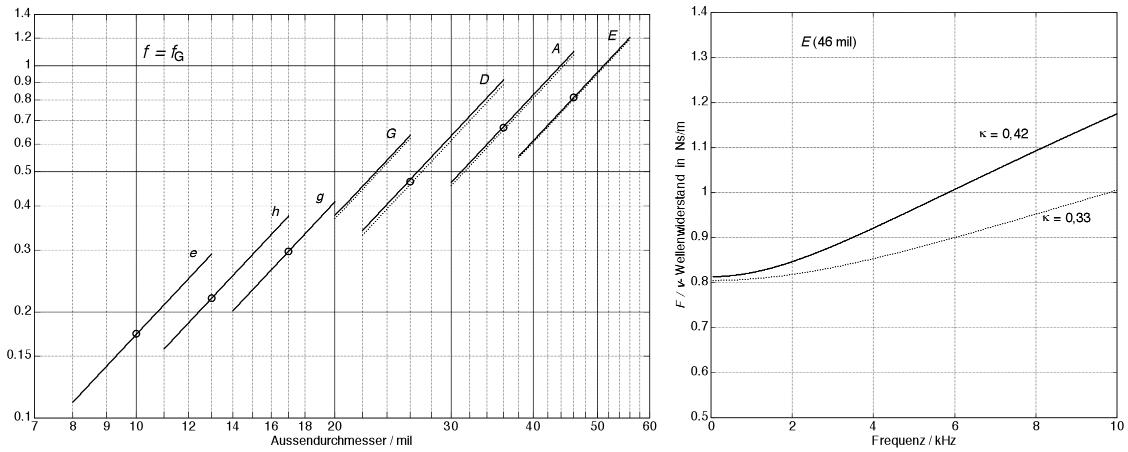

Fig. A.5.1: Wave impedance for different outer diameters. The dashed curves belong to strings with a relatively thin core. In the E2 string (right), the influence of bending stiffness can be seen in the upper frequency range. With decreasing string diameter, the bending stiffness loses its importance; the flexural wave becomes a transversal wave. "Aussendurchmesser" = outer diameter, "Frequenz" = frequency; "h" (string) = B (string)

Taking into account the **flexural rigidity**, we encounter more complicated relationships. The wave equation now contains, in addition to the second spatial derivative, a fourth spatial derivative. Because of this, not only progressive waves (in both directions) occur, but also exponentially decreasing fringe fields (in the vicinity of the bearings). The progressive waves need to be classified into bending waves and bending-moment waves, and therefore it is necessary to define an M/w-wave-impedance in addition to an F/v-wave-impedance. The M/w-wave-impedance similarly connects the bending moment to the angular velocity. Both resistances are real, but frequency dependent. In a string, the transverse dimensions are small compared to the wavelength of the flexural wave ("thin rod"), and a simplification may be applied: outside of the fringe fields that extend merely a few millimeters, the description for *one single* wave type is sufficient. The four wave quantities are: F, v, M, w (Chapters A.4.1 & A.4.2); given two quantities, the other two may be calculated. **Fig. A.5.1** shows the F/v-wave-impedance for the fundamental frequencies of the strings. The ratio of core-to-outer-diameter (κ) has little effect for $f = f_G$; for heavy strings, and high frequencies, the bending stiffness needs to be considered, after all.

♣ In chapter A.5, M stands for a mechanical moment, and M for the scale length of the strings.

Taking into account the bending stiffness, the *F*/*v*-wave-impedance is calculated as:

$$Z_{WF} = \frac{F}{v} = \frac{\Psi + B \cdot k^2}{\omega / k} = \frac{\Psi + B \cdot k^2}{c}; \qquad k = \pm\sqrt{\frac{1}{2B}\left(\sqrt{\Psi^2 + 4Bm'\omega^2} - \Psi\right)}$$

The sign of the wave impedance depends on the direction of propagation of the wave: for waves travelling to the right (increasing *z*), Z_{WF} is positive; it is negative for waves travelling to the left. The flexural stiffness changes the *F*/*v*-wave-impedance in two ways: the summand Bk^2/c is added, and the first term is also changed because the phase velocity ($c = \omega/k$) increases with increasing frequency. **Fig. A.5.2** illustrates, for an E_2-string, the influence of the bending stiffness on the *F*/*v*-wave-impedance. In the range of low frequencies, Bk^2 can be neglected with respect to Ψ, with the wave impedance approximately depending only on Ψ/c (for conversions see above). At higher frequencies, the influence of bending stiffness on the treble strings is small (G-string in Fig. A.5.2), but for the bass strings it is pronounced (E_2-string in Fig. A.5.2) – in particular given a relatively thick core (i.e. large κ).

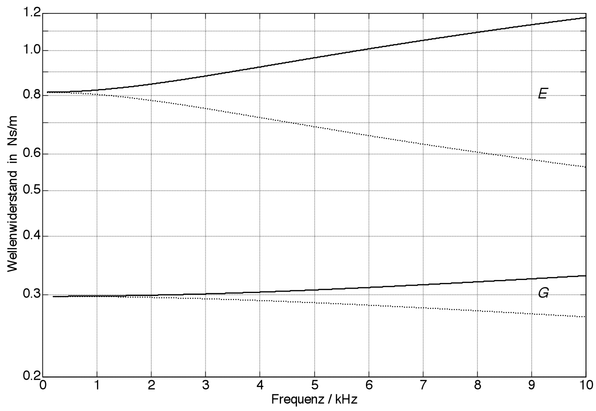

Fig. A.5.2: Influence of bending stiffness on the *F*/*v*-wave-impedance. E2-string (46 mil, $\kappa = 0.42$), G-string (17 mil, plain). The solid line indicates the total impedance, the dashed line indicates the first term ($\Psi k/\omega = \Psi/c$).
"Wellenwiderstand" = wave impedance,
"Frequenz" = frequency

Taking into account the bending stiffness *B*, we have a 4th-order differential equation: independently of transverse force and transverse string displacement, excitation with a moment or a rotational movement is possible, as well, and the string end may be (at least theoretically) free, supported, clamped, or guided. The idealized boundary conditions for the force *F*, the velocity *v*, the angular velocity *w*, and the moment *M* result in:

free: $F = 0$, $M = 0$; supported: $v = 0$, $M = 0$; clamped: $v = 0$, $w = 0$; guided: $F = 0$, $w = 0$.

The free end of the string will be called into question immediately: it cannot exert any clamping force. Even in the theoretical literature, a guided mounting is listed only for the sake of completeness. However, it must not be overlooked here that these bearing conditions (bearing impedances) are frequency-dependent. At $f = 0$ Hz a tensioning force is indispensable, but at $f \neq 0$ entirely different conditions can occur, as the following example shows: a spring-loaded mass is defined as the string bearing; the bearing impedance thus calculates as: $Z = j\omega m + s/j\omega = (s - \omega^2 m)/j\omega$. For $f = 0$, this bearing acts like a spring – it can absorb static tensioning forces. At resonance, however, the impedance is zero - which implies: no force, despite movement.

In addition to the *F/v*-wave-impedance, the *M/w*-wave-impedance must also be taken into account for the rigid string. The *M/w*-wave-impedance connects the moment *M* to the angular velocity *w*. This angular velocity is not the angular frequency ω with which the string vibrates, but the plucking-attack-dependent rotational velocity of the individual string particles (Fig. A 4.1).

$$Z_{WM} = \frac{M}{w} = \frac{B \cdot k}{\omega} = \frac{B}{c}$$ *M/w*-wave-impedance

In the range of low frequencies, the **phase-velocity** *c* can be calculated in good approximation from the basic string frequency f_G and twice the scale length *M*: $c = 2M \cdot f_G$. Since the bending stiffness depends on the string diameter to the power of four, Z_{WM} also increases with the string diameter to the fourth (**Fig. A.5.3**). Given increasing frequency, *c* may however no longer be taken to be constant; rather, an increase over *f* must be considered, especially in the case of the bass strings (**Fig. A.5.4**):

$$c = \omega \cdot \sqrt{\frac{2B}{\sqrt{\Psi^2 + 4Bm'\omega^2} - \Psi}}$$ Phase velocity

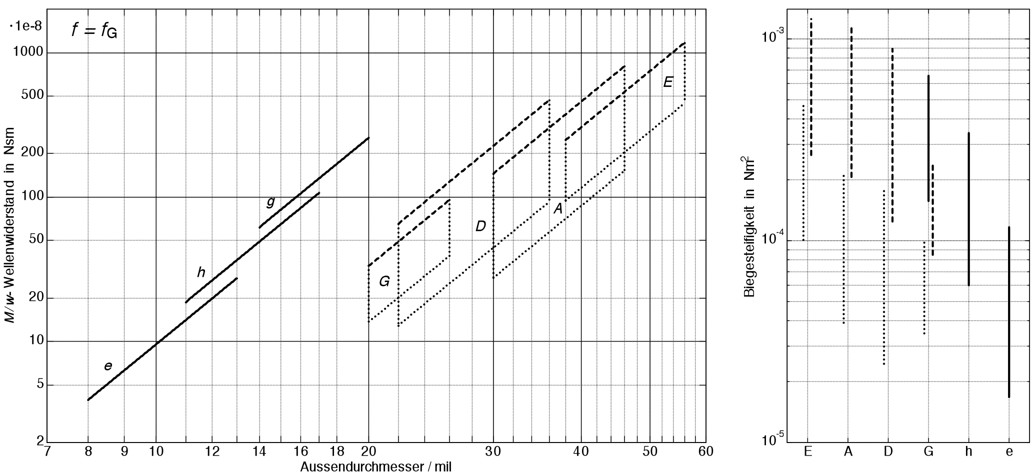

Fig. A.5.3: *M/w*-wave-impedance (left), bending stiffness (right). Solid strings (**---**), wound strings with thick core (**-----**),wound strings with thin-core (**.......**). “Wellenwiderstand” = wave impedance; “Biegesteifigkeit” = bending stiffness; “Aussendurchmesser” = outer diameter. “h” (string) = “B” (-string).

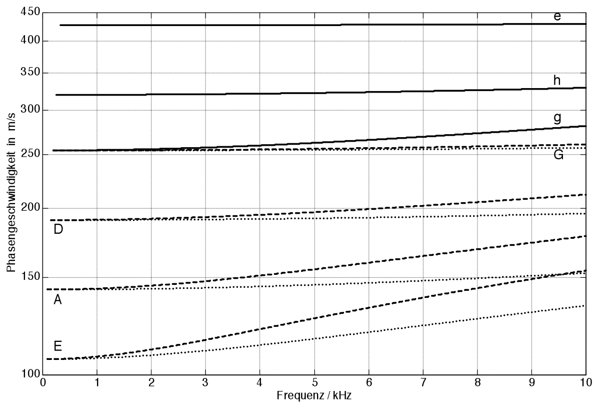

Fig. A.5.4: Phase velocity of bending wave: solid strings (**---**),
wound strings with thick core (**-----**),
wound strings with thin core (**.......**).
“Phasengeschwindigkeit” = phase velocity
“Frequenz” = frequency;
“h” (string) = B (-string)

For **Dilatational waves**, the wave impedance Z_W is also calculated from the product of length-specific mass m' and phase velocity c, but now the phase velocity of the Dilatational wave is to be assumed as follows:

$$Z_W = \frac{F}{v} = m' \cdot c = S \cdot \sqrt{E \cdot \rho}$$ Wave impedance (Dilatational wave)

Here, S stands for the cross-sectional area, E for Young's modulus, and ρ for the density. Since no dispersion occurs, the wave impedance is frequency-independent.

For wound strings, again the ratio of core diameter / outside diameter needs to be considered: $\kappa = D_K / D_A$. The winding increases the mass without significantly increasing the longitudinal stiffness (in approximation). The characteristic impedance results in:

$$Z_W = S_A \cdot \kappa \cdot \sqrt{E \cdot \bar{\rho}}\,; \qquad \bar{\rho} \approx 0{,}9 \cdot \rho$$ Dilatational-wave-impedance for wound strings

The impedance of the Dilatational-wave-resistance is about twenty times as large as the impedance of the transverse wave.

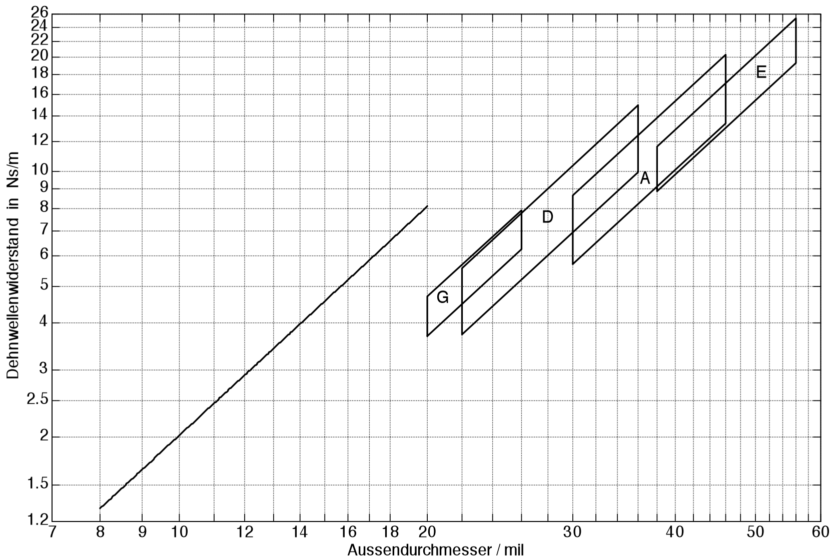

Fig. A.5.5: Wave-impedances for Dilatational-wave propagation. "Dehnwellenwiderstand" = Dilatational-wave-impedance; "Aussendurchmessen" = outer diameter.

