### 7.5.3 The mechanical bridge-impedance

Interpreting string and saddle (bearing of the string)♣ as parts of a mechanical system, two different paths offer themselves towards a mathematical description: either the space- and time-dependent analysis including differential- and wave-equations, or the spectral presentation of the post-transient, settled condition. The previous chapter had analyzed the processes in terms of their evolution over time, below we now get into the frequency domain.

For the simple model of the string, the saddle is motionless and reflects ideally, with reflection-factor $r$ emerging to be +1 (force) or -1 (motion), respectively. From the real wave-impedance $Z_W$ of the string, and the complex saddle-impedance (bearing impedance) $Z_L$, $r$ had been calculated in Chapter 2.5. To keep the relations manageable, we shall look at a single wave-type (e.g. the transversal wave), and a loss-free bearing. The saddle-impedance $\underline{Z}_L = \underline{F}/\underline{v}$ is now a **two-pole function of the reactance**. The reactance is the imaginary part of a complex impedance. A reactance two-pole includes an imaginary impedance only, and therefore only masses and springs are allowed as elements – damping resistances are not. Given the complex frequency $p$ [e.g. 6], the mass impedance is calculated as $pm$, and the mass reactance as $\omega m$, respectively; the spring impedance is $s/p$, and the spring reactance is $-s/\omega$. If a mechanical reactance two-pole is comprised of masses and springs (loss-free, there will be no other elements), its impedance is given by a reactance two-pole function of the form:

$$\underline{Z}(p) = \frac{z_0 + z_2 p^2 + z_4 p^4 + \ldots + z_\nu p^\nu}{n_1 p^1 + n_3 p^3 + \ldots + n_\mu p^\mu} \qquad \text{Reactance-two-pole function}$$

Herein, $z_i$ are the coefficients of the numerator, and $n_i$ those of the denominator. The largest power in the numerator-polynomial ($\nu$), is either larger by 1 than that of the largest denominator-polynomial ($\mu$), or smaller by 1. The larger value of $\nu$ and $\mu$ corresponds to the **order** $n$ of the system – in canonical systems it would be the number of free memories.

**Example:** a system containing but *one single* spring is a 1st-order system. It would already be possible to describe an ideal bearing that way, but the spring-stiffness would then need to be infinite (a spring of infinite stiffness is unyielding). Given finite stiffness, an approximation is possible as long as the bearing reactance remains large relative to the wave-impedance. With $s = 10^6$ N/m we get, at 1 kHz, a bearing reactance of –160 Ns/m; this may be seen (in terms of the absolute value) already as large compared to the wave impedance of a steel string (0,1 ... 1 Ns/m). A bearing with this kind of suspension would reflect low-frequency waves almost exactly as a rigid bearing would. Mounting a small mass ahead of this bearing spring, and another spring ahead of this small mass, a 3rd-order system would emerge. At low frequencies, this new system would behave in a spring-inhibited manner, and at high frequencies the same. Between the two resonance frequencies, however, it would be mass-inhibited (inert). The frequency dependency of the bearing-reactance has two effects: a de-tuning of the frequencies of the partials, and the generation of additional partials that would not exist with an ideal string bearing (Chapter 2.5.2). The higher the order of the bearing impedance, the more additional partials are created.

What is the typical order of magnitude of a bearing-impedance? It should be infinite, since nut, bridge and body are continua – but in practice it is finite, after all, because we regard merely a finite frequency range.

♣ As already noted on page 7-21, the term „saddle“ is used here generally for the bearing of the string, i.e. for both nut and bridge – and for the respective fret, as well, in case of a fretted string.

The theorems about reactance-two-pole functions [e.g. 7] say that along the $j\omega$-axis, poles and zeroes alternate. Between the poles and zeroes, the mechanical reactance-two-pole (i.e. the "reactive" string bearing) behaves either like a spring or like a mass; the partials of the string are correspondingly detuned higher or lower (Chapter 2.5.2). One pole-zero-pairing each (on the positive imaginary axis) generates an additional partial; given $n = 8$ we therefore get already 4 additional partials. Since all parts of the guitar are force-fitted to each other, we would in theory have to account for a whole lot of vibration-happy sectional masses and springs, and consequently would have to deal with a high number of additional partials. That is for the loss-free bearing, though! As soon as we grant **resistive elements** to the bearing, the situation changes from the ground up: only those resonances that are extremely weakly damped can change the phase by 2π, and generate additional tones. All other resonances only result in small frequency shifts.

The mechanical impedance of a lossy bearing is not a reactance-two-pole function but a **two-pole function**, i.e. a real, rational and positive function of $p$. All poles and zeroes of the bearing impedance $\underline{Z}(p)$ are located left of the $j\omega$-axis. Mapping $\underline{Z}(p)$ onto the complex reflection factor $\underline{r}(p)$, we do not obtain a pure all-pass function – rather, phase *and damping* are frequency dependent.

$$\underline{r} = \frac{W - \underline{Z}}{W + \underline{Z}} = \frac{W - \underline{Q}/\underline{V}}{W + \underline{Q}/\underline{V}} = \frac{W \cdot \underline{V} - \underline{Q}}{W \cdot \underline{V} + \underline{Q}}$$

Complex *v*-reflection-factor

The bearing impedance can be expressed as a rational function $\underline{Q}$ / $\underline{V}$; from the $n$ poles and zeroes, the mapping generates $n$ new poles and zeroes – the order $n$ is retained (for real wave-impedances). From $W \cdot \underline{V} + \underline{Q}$, we obtain via zeroing the poles of the reflection factor – all positioned left of the $j\omega$-axis (stable system). From $W \cdot \underline{V} - \underline{Q}$, we obtain the zeroes of the reflection factor, but these may now be distributed across the whole of the $p$-plane! If $r$ zeroes are *located on the jω-axis*, **matching** will occur at the corresponding frequency: the reflection-factor is zero – the bearing absorbs the whole of the wave energy. If $r$ zeroes are *located right of the jω-axis*, the reflection factor contains (inter alia) an **all-pass**. If $r$ zeroes are *located left of the jω-axis*, the reflection factor is **without an all-pass** (= of minimal phase).

Slightly simplifying: each resonance of the bearing results in a pole/zero-pair of the reflection factor. The poles of the reflection factor are always located within the left $p$-half-plane; the zeroes of the reflection factor may be located left or right. A zero on the left merely causes the detuning of a partial, while a zero located on the right will generate also an additional partial.

A resonance circuit (i.e. a spring/mass/damper-system) within the bearing will make for a narrow-band absorption of vibration energy; it decreases the magnitude of the reflection factor from 1 to values just shy of 1. If the resonance circuit is of minimum phase, there will only be a small phase shift: below the resonance, the phase of the reflection will be slightly negative while above the resonance, it will be slightly positive. With increasing distance from the resonance frequency, the phase shift decreases towards zero. However, in case the resonance circuit contains an all-pass (i.e. it is not of minimum phase), it will shift the phase with increasing frequency by -2π, generating an additional partial. The basics for this approach to looking at things are found in systems theory [6, 7]; the reflection process can be understood as mapping characteristic of a linear and time-invariant system.

Let us take a little detour to explain the process of a reflection for an electric transmission line – subsequently we shall then look at the corresponding analogy for the mechanical line. An **electric transmission line** with a wave-impedance of $W$ = 50Ω [5] is terminated with an RLC-series-circuit, for example $R$ = 25Ω, $L$ = 0.1H, $C$ = 1μF. A reflection factor $\underline{r}$ follows:

$$\underline{r} = -\frac{R + pL + 1/(pC) - W}{R + pL + 1/(pC) + W} = -\frac{p^2L + (R - W)p + 1/C}{p^2L + (R + W)p + 1/C}$$

The zeroes of the numerator-polynomial are located in the right-hand $p$-half-plane, i.e. the reflection is not of minimum phase but includes an all-pass. If we modify the three elements of the termination-impedance to be $R$ = 100Ω, $L$ = 0.2H, $C$ = 0.5μF, this will not change the magnitude of the reflection factor. The phase, however, does change: a minimum-phase (i.e. all-pass-free) system results. **Fig. 7.35** depicts the corresponding magnitude, phase, and locus of the reflection factor. For minimum-phase, the $\underline{r}$-locus runs left of the coordinate-origin, while for the solution comprising an all-pass, the solution encircles the origin.

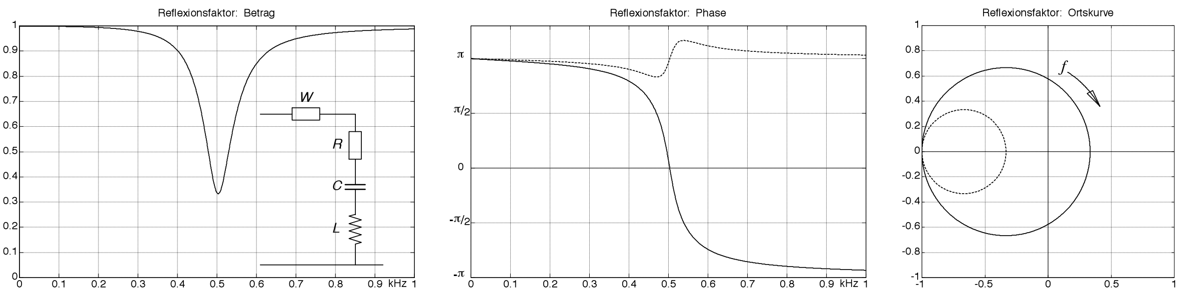



**Fig. 7.35:** Reflection factor ("Reflektionsfaktor"): frequency response of magnitude ("Betrag") and phase, and locus ("Ortskurve"); ---- = minimum-phase.

Now on to the mechanical transmission line: the **string**. Given a dispersion-free situation, the wave impedance is again real, and the termination (bearing) impedance is a two-pole-impedance. As an example, a 5th-order bearing consists mainly of a stiff spring but also includes two small masses, two springs and four dampers (**Fig. 7.36**). It is not directly obvious whether or not the mechanical system generates an all-pass-free reflection; both characteristics may result from the same structure – merely the component-values differ.

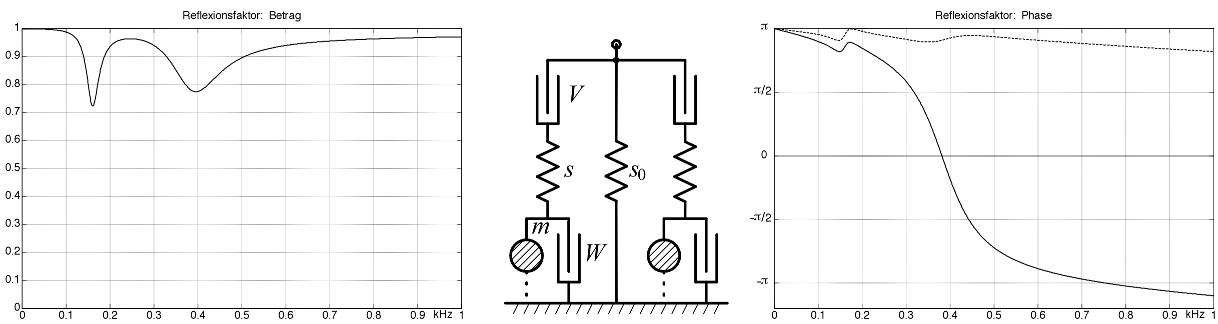


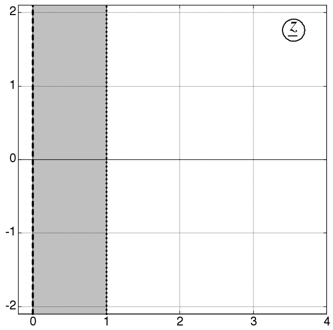

**Fig. 7.36**: Magnitude and phase of the reflection factor ("Reflektionsfaktor") for two sets of component values. The frequ. responses of the magnitude ("Betrag") are identical, those of the phase differ (---- = minimum-phase).

| $V$ | $s$ | $m$ | $W$ | $s_0$ | $V$ | $S$ | $m$ | $W$ |
|---|---|---|---|---|---|---|---|---|
| 3,98 Ns/m | 2859 N/m | 0,36 g | 0,1 Ns/m | 5087 N/m | 19,1 Ns/m | 2054 N/m | 1,43 g | 0,17 Ns/m |
| 25,6 Ns/m | 7498 N/m | 0,33 g | 0,005 Ns/m | 2221 N/m | 34,9 Ns/m | 281 N/m | 0,244 g | 0,048 Ns/m |

**Table:** Component values. The last line results in a reflection comprising an all-pass.

From the point of view of function-theory, mapping the complex $\underline{Z}$-plane onto the complex $\underline{r}$-plane is a **conformal mapping**. Normalizing the bearing impedance $\underline{Z}$ relative to the wave-impedance $W$, we obtain – with $\underline{z} = \underline{Z}/W$ – the following normalized conformal mapping:



$$\underline{r} = \frac{W - \underline{Z}}{W + \underline{Z}} = -\frac{\underline{z} - 1}{\underline{z} + 1}$$

$$\underline{z} = \frac{\underline{Z}}{W} = \frac{1 + \underline{r}}{1 - \underline{r}}$$

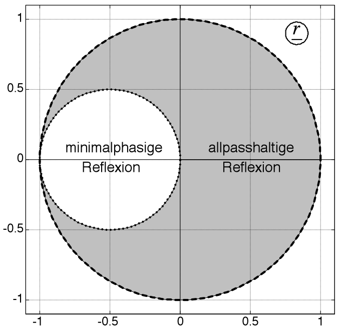



**Fig. 7.37:** Conformal mapping of the normalized $\underline{Z}$-plane onto the $\underline{r}$-plane.
"Minimalphasige" = minimum phase; "Allpasshaltige" = comprising an all-pass; "Reflexion" = reflection;

The unit-circle of the $\underline{r}$-plane is a mapping of the imaginary axis of the $\underline{z}$-plane – the left-hand $\underline{z}$-plane is mapped to the exterior of the unit circle. Because all real bearing-impedances are two-pole functions with a non-negative real part, the magnitude of the reflection factor cannot become larger than 1 (this is mandatory from an energy point-of-view, as well). The band of the $\underline{z}$-plane hatched in grey is mapped to the grey area of the $\underline{r}$-plane; the range of Re($\underline{z}$) > 1 is mapped onto the pale-ish/thin circle. For Re($\underline{z}$) > 1, Re($\underline{z}$ – 1) > 0 holds, i.e. it is a two-pole function including zeroes in the left-hand $\underline{r}$-plane (thus of minimum phase). In the case of the electrical transmission line (the above example), $\underline{z}$ is the straight line $\underline{z}$ = 0.5 for $R$ = 25 Ω. It is located in the grey area – the reflection therefore comprises an all-pass. For the mechanical line (string across a bearing), **Fig. 7.38** shows the loci – including one peculiarity:

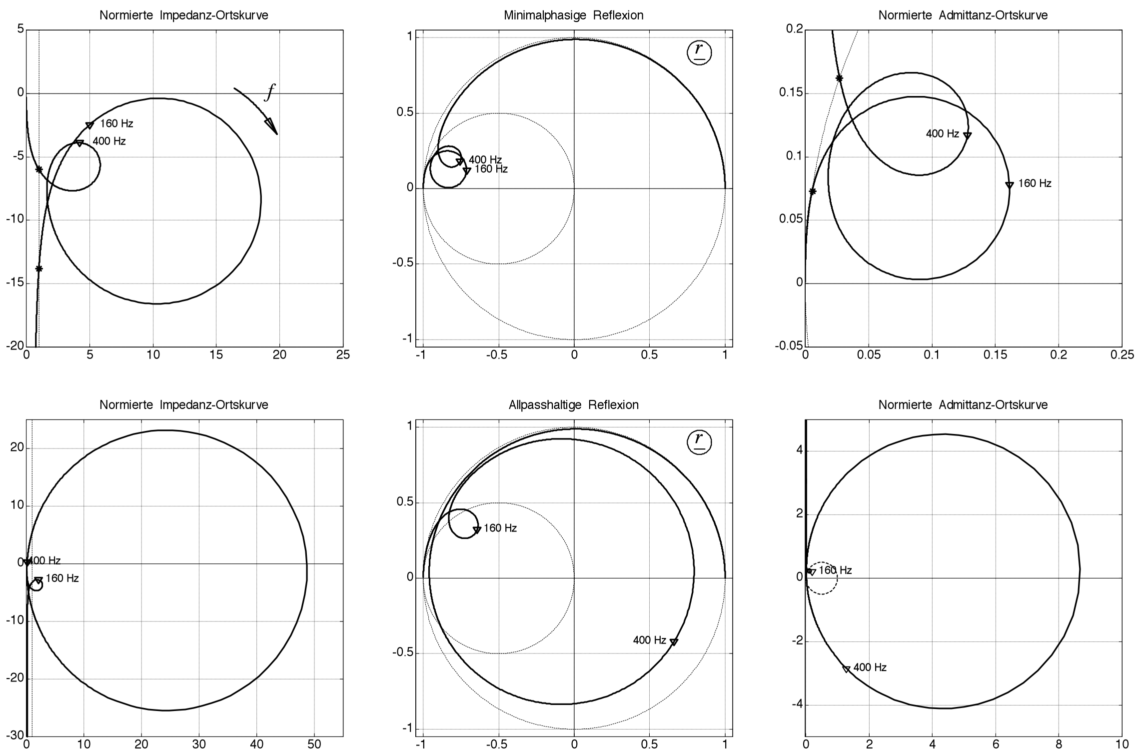



**Fig. 7.38:** Loci for the mechanical line; compare to Fig. 7.36. All loci are run through clock-wise with increasing frequency. "Normierte" = normalized; "Ortskurve" = locus; "Impedanz" = impedance; "Minimalphasige" = minimum phase; "Allpasshaltige" = comprising an all-pass; "Reflexion" = reflection; "Admittanz" = admittance.

Between 100 Hz and 650 Hz, the normalized impedance-locus runs *to the right* of the dashed delimitation line; the corresponding reflection factor lies within the small circle. The two maxima of the reflection damping (at 160 Hz and 400 Hz) cause two small loops in the $\underline{r}$-locus – they are located within the small circle and therefore have minimum-phase characteristic. Globally seen, though, the phase of the *v*-reflection-factor changes from π to 0, which is a characteristic of every spring-type bearing ($s_0$). For frequencies approaching zero, the system shown in Fig. 7.36 acts spring-like; the impedance thus shows a pole at $p = 0$ (and a zero at $p = \infty$). If we wanted to avoid this peculiarity, the bearing would have to be designed to have an essentially resistive characteristic (i.e. it would have to be a damper); however, this setup would not enable the system to absorb any pre-load force. Therefore we have a spring-type bearing, and consider all-pass characteristics only within the relevant frequency range.

Fig. 7.38 also contains the locus of the normalized **admittance** $\underline{Y} = G + jB = 1/\underline{Z}$. The real part $G$ of the admittance is termed **conductance**, the imaginary part $B$ is called **susceptance**. Whether you will want to work with the impedance (and its components resistance and reactance), or with the admittance (and its components conductance and susceptance) is a matter of taste; the conversion from one world to the other is simple. When calculating the absorption in a bearing, the admittance yields the shorter formula – that's why it will be used in the following. The power absorbed in the bearing is not available anymore to the reflected wave – every bearing will cause, besides phase shifts, an absorption (i.e. damping). The **degree of absorption** $a^2$ tells us the relative portion of the effective power irreversibly absorbed in the bearing:

$$a^2 = 1 - r^2$$

$a^2$ = degree of power-absorption, $r^2$ = degree of power reflection.

Both these magnitudes depend on the corresponding factor with a square-relationship: if, for example, the reflection factor is $r = 50\%$, then 25% of the power gets reflected, and 75% gets absorbed. The **effective power** $P_W$ absorbed by a two-pole may be represented in four ways:

$$P_W = \tilde{F}^2 \cdot G = \tilde{v}^2 \cdot R = \tilde{F}^2 \cdot \frac{R}{(R + jX)^2} = \tilde{v}^2 \cdot \frac{G}{(G + jB)^2}$$

Effective power

For the string bearing, the degree of absorption in the bearing computes as:

$$a^2 = \frac{4 \cdot WG}{(WG + 1)^2 + (WB)^2} \approx 4 \cdot WG$$

Degree of power-absorption in the bearing

Assuming a relatively stiff bearing with small conductance and small susceptance, the power absorption is proportional to the conductance. Measurements carried out by Fleischer [2006] show that at least the conductance mostly remains below 0,01 s/kg – which is small compared to the inverse of the wave-impedance of customary strings (1 – 10 s/kg). Fleischer does not explicitly specify measurements regarding the susceptance, but the order of magnitude is comparable for circle-shaped loci. In the example presented in Fig. 7.38, the conductance in the minimum-phase system reaches values just above 0,15 s/kg; however, the absorption is chosen to be untypically large in order to be able to depict the curves purposefully. For the reflection comprising an all-pass, however, a rather different scenario emerges: here we get conductance-values that are larger than the inverse of the wave-impedance. This is due to the longitudinal (dilatational) waves already mentioned above – almost half of the power of the transversal waves arriving at the bearing they can be converted into them (Chapter 7.5).

The (at least theoretically) high importance of the presence of an all-pass is also shown by the following measurement that had already been indicated in Fig. 7.34. In **Fig. 7.39,** we see a segment from the spectrum of a string tuned to 152 Hz. The vertical grid-lines are matched to the calculated frequencies of the partials as they would be present in a rigid string clamped fixedly at its ends. Thus the spreading of partials caused by the bending stiffness is considered here – the correspondence remains rather poor, though: the frequencies of 8 of the partials clearly miss the calculated values, und there are 9 additional lines. Both the deviations and the generation of additional partials are the result of the **phase of the reflection-factor**: minimum-phase zeroes cause **detuning**, all-pass-behavior generates **additional tones**.

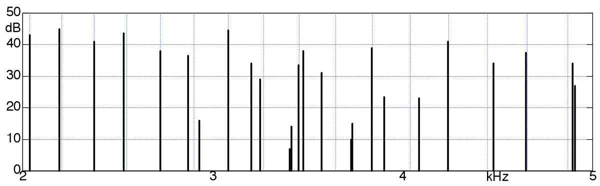



**Abb. 7.39:** Spectrum of a plucked string running across a bearing saddle via a 45° bend angle.

We must, however, not imagine the sound-effect of the **additional tone** as an inharmonic interference next to the actual guitar sound. If the level of such an additional tone is small, it remains totally inaudible. Quantitatively, it is difficult to state anything here because the psycho-acoustic masking mechanisms are highly complicated for complex stimuli. A qualitative statement is easy to formulate: in every guitar sound, there are partials that are visible in the spectrum but still remain inaudible. If they do become audible (given sufficient level), they come across not as interference but as a change in sound color. For example, an additional 3416-Hz-tone appearing next to a 3406-Hz-tone may cause a beating effect in this frequency range. However, since inharmonic (spread out) spectra sound with a slight beating effect anyway, the addition of an extra tone will at most make for a marginal change (as long as the additional tone stays within certain limits). It is not possible to quantify this statement further for your typical situation on stage (or in the studio) because there are too many unknown influences: filters, amplifiers, loudspeakers, room resonances, etc.

Is it then purposeful at all to measure guitar-vibrations, since at the end there is (yet) no way to give quantitative statements about the sound? Of course, measurements can only be supplementary to listening experiments, and not a replacement. Measurement of vibrations support (or refute) assumptions about models – and they therefore deliver building blocks for a psycho-acoustical model about sound. This model at present only exists in rudimentary form but takes shape as the findings progress. From a theoretical point of view, the exact understanding of reflection processes naturally is indeed important: it allows for defining reasons for abnormalities even if the latter do not become audible in every case. It is reassuring to be able to in fact attribute an unexpected result to the investigated object, rather than fearing that the equipment is at fault. We could call such a fault a "system-immanent artifact", but it would be disturbing just the same. Spectral analysis – as the basis for every determination of partials – includes many **artifacts** that could massively influence the final result. The spectrum calculated according to the classic Fourier-integral does not exist at all for real tones: not many people are willing to wait the formally required infinite period of time. Weighting windows create approximations within a finite time, but they do this at the expense of un-ambiguity. Chapter 7.6 will address, in depth, the instrumentation analytics, but first let us take a look at conductance- and absorption measurements for real guitars – given all the above theory it will be good to now show some quantitative measurement results, as well.