1 Basics of the Vibration of Strings

As a stringed instrument, the guitar belongs to the subgroup of composite chordophones/lute instruments/crossbar instruments. The strings form frequency-determining oscillators; they radiate their vibration either directly as airborne sound or – after conversion into an electrical signal by the pickups – via the guitar amplifier. Being a mechanical oscillator, the string is briefly fed energy by a plucking action … not a lot of energy but enough to entertain an auditorium even without an amplifier. It would actually be possible to heat up one liter of water to boiling temperature using this plucking energy: to achieve this objective, the guitarist would have to pick the string about 60.000.000 times. That sounds worse than it is – picking the string 5 times per second it would take about 2 years if we assume that no break is taken, and that the heat-insulation is perfect. Old Sisiphus would be happy to "enjoy" such working conditions. Admittedly, approaching the topic of producing art from a mechanistical/ operationistical angle receives ambivalent assessment from the involved research disciplines. Elementartistic schools of though have to put up with being by the gestalt-psychologists that the whole is more than the sum of its parts, after all. It doesn't really help to counter the insight "Hendrixian genius is more than pure superposition of vibrations" with the existentialistic appearing question "yeah well – and where is he now?" … all too different are the doctrines. The following considerations therefore target exclusively vibration mechanics – as a part of the whole … as an essential part of the whole.

Translator's remark: *in this chapter, often the bridge and the nut of the guitar are taken as the points between which the guitar string vibrates. Of course, all basic considerations apply to the fretted string in the same way – the string then vibrates between bridge and fret. This is not always explicitly indicated, and therefore the term "nut" should be considered to appropriately include the term "or fret", as well.*

1.1 Transversal waves

The strings of an electric guitar are made of steel, with its **density** ρ just below $8 \cdot 10^3\ \mathrm{kg/m^3}$. A steel string with a **diameter** D is stretched to the **length** L by applying the **tension force** Ψ. Fretting the string on the fretboard shortens the length. Typical lengths are just shy of 65cm for the open (unfretted) string (= scale M). Plucking the string (with a finger or a pick) displaces the string in the transversal direction; subsequently there is a free, damped vibration. After the plucking-release, a transversal motion (transversal wave) propagates from the plucking position in both directions of the string. The **propagation speed** c of this wave running along the string is (with $\rho = 8 \cdot 10^3\ \mathrm{kg/m^3}$):

$$c = \frac{2}{D}\sqrt{\frac{\Psi}{\pi\rho}} = \frac{\sqrt{\Psi/\mathrm{N}}}{D/\mathrm{mm}} \cdot 12{,}6\,\frac{\mathrm{m}}{\mathrm{s}}$$

Propagation speed

Given a string of a diameter of 0,35 mm and a tension force of 50 N, c calculates to 255 m/s. However, this propagation speed (in the direction of the string) must not be confused with the velocity that the string oscillates back and forth with in the transverse direction. To avoid any confusion, the transverse velocity is termed **particle velocity** v. More detailed investigations reveal that c is not constant but depends on the frequency (dispersion); more about this in Chapter 1.3.

Moving with the propagation speed, the transversal wave runs off in both directions and is reflected at both ends (nut and bridge, respectively). As a reflection, it then returns to the point of origin. We may imagine and model the process of reflection as a superimposed signal originating from a **mirror source** positioned behind the end of the string (**Fig. 1.1**). In this model, the primary wave excited by the plucking runs beyond the end of the string (i.e. it is not reflected), but an additional superimposed (added) mirror wave runs opposed to the primary wave. At the fixed end of the string, both waves meet. It is obvious that the displacement of the mirror wave needs to be in **opposite phase** to the primary wave such that the end of the string indeed remains (ideally) at rest and immobile. This phase reversal is valid at both nut and bridge in the same way.

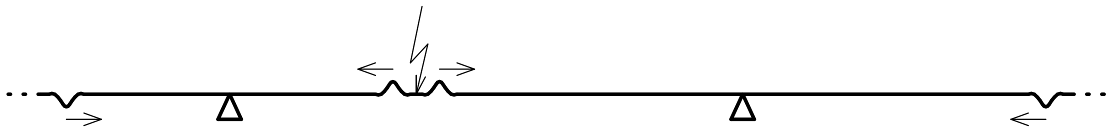

Fig. 1.1: Propagation of a transversal wave on a clamped string.

As the reflections arriving from nut and bridge reach the origin-point of the plucking, they continue further, are then reflected again at the respective other end of the string, and run back to the plucking point with the original phase. Arriving there after having covered $2L$, one full **period of the fundamental oscillation** T has passed. The reciprocal of T is the **fundamental frequency** f_G of the string. A steel string of a length of 0,65 m and a diameter of 0,35 mm oscillates – at a tension force of 50 N – with a fundamental frequency of 196 Hz (note G_3).

The frequencies of the open strings (in regular tuning) are **E** = E_2 = 82.4Hz, **A** = A_2 = 110Hz, **d** = D_3 = 146.8Hz, **g** = G_3 = 196Hz, **b** = B_3 = 246.9Hz, and **e'** = E_4 = 329.6Hz.

The fundamental frequency of the string depends on the **tension force** Ψ, the **density** ρ, the **diameter** D, and the **length** L. Quadrupling the force, or halving the length, or halving the diameter, respectively, doubles the fundamental frequency:

$$f_G = \frac{c}{2L} = \frac{\sqrt{\Psi/\pi\rho}}{DL} = \frac{\sqrt{\Psi/\mathrm{N}}}{D/\mathrm{mm}\cdot L/\mathrm{m}} \cdot 6{,}3\,\mathrm{Hz}$$

Fundamental frequency

The tension force Ψ required to obtain a certain fundamental frequency calculates based on the length L of the string, and on the material data of the density ρ and the diameter D. Fundamental frequency and string-length appear as a product; given a string tensioned with a constant force, fundamental frequency and string length are therefore reciprocal to each other:

$$\Psi = (f_G \cdot L)^2 \cdot \pi\rho D^2 = c^2 \cdot \pi\rho D^2 / 4$$

Tension force

Because the actual oscillation processes are rather complicated, idealizing models are employed. In the simplest case, planar polarization, frequency-independent propagation speed, absence of losses, and ideal reflections are assumed. The string is described as a linear, time-independent LTI-system.

The periodic repetition caused by the reflections can be seen as a (temporal) convolution of the excitation impulse with a causal Dirac-pulse. Causal means that the signal is zero for the negative time axis. A causal Dirac-pulse contains equidistant Dirac-impulses for $t \geq 0$. A temporal convolution corresponds, in the spectral domain, to a multiplication of the excitation spectrum with the spectrum of the causal Dirac-pulse. This latter spectrum necessarily is complex, since the time-function (causal Dirac-pulse) is neither odd nor even (mapping theorem). Using partial fraction decomposition, it can be shown that a co-tangent-shaped spectrum of the imaginary part is linked to the causal Dirac-pulse; the spectrum of the real part is a spectral Dirac-comb. This complex spectrum would have to be multiplied by the excitation spectrum – however this is still too complicated for most considerations.

For this reason, further idealization is in order. The (un-damped) oscillation is not induced at $t = 0$ but continues from the infinite past to the infinite future. The period of the oscillation may be developed into a Fourier series since it is in a steady state with regard to its periodicity. A **line spectrum** results as the spectrum of the oscillation, with the frequency lines at the integer multiples of the fundamental frequency.

This way, the overall oscillation can be seen as the sum of superimposed (added) single tones – they are called partials or (because of the integer frequency relations) **harmonics**. The fundamental is the 1st harmonic, with the 2nd harmonic located at double the frequency of the fundamental. In music, the 2nd harmonic is called the 1st **overtone**. This terminology extends to the higher harmonics correspondingly (3rd harmonic = 2nd overtone, etc.).

Reality differs considerably from these idealizations. A line spectrum requires a periodic signal of infinitely long duration. In signal theory, the term ‘periodic’ implies that a certain section of the signal is infinitely repeated in identical shape. However, as it oscillates back and forth, the string looses energy, and therefore an identical repetition of any signal section is not possible. The oscillation of the string therefore is a non-periodic signal that has no actual line spectrum affiliated to it; rather, the spectral lines are broadened into funnels due to the damping. The reasons for the energy loss are **dissipation** and radiation: the motion energy in the string is partially converted directly into heat, and partly radiated as sound-energy. The frequency dependent propagation speed (**dispersion**) – discussed more extensively in Chapter 1.3 – constitutes an additional effect that must not be ignored for more detailed investigations.

Even though the string oscillation is in fact of dispersive and dissipative character, it is still purposeful for the understanding of the motion processes to use a simplified, idealized view. This holds in particular as long as we only regard short sections of the time signal.

An idealized plucking will displace the string triangularly (**Fig. 1.2**). After the pick (or the finger) has lost contact to the string, the latter will ideally oscillate freely and without damping. The shape of the lateral displacement can be seen as superposition of two partial waves running in opposite directions. Both partial waves are identical at the moment of plucking but run away from each other in opposite directions for $t > 0$; the magnitudes of both propagation velocities are equal. For $t = 0$, the displacement of each partial wave at the nut and the bridge is zero; it is at its maximum value at the plucking location. The triangular shape continues in a point-symmetrical (odd) manner at the nut and the bridge as mirror wave. The displacements of both *partial waves* are superimposed to yield the displacement of the string. The same holds correspondingly for all derivatives, e.g. for the propagation speed.

 Translation by Tilmann Zwicker

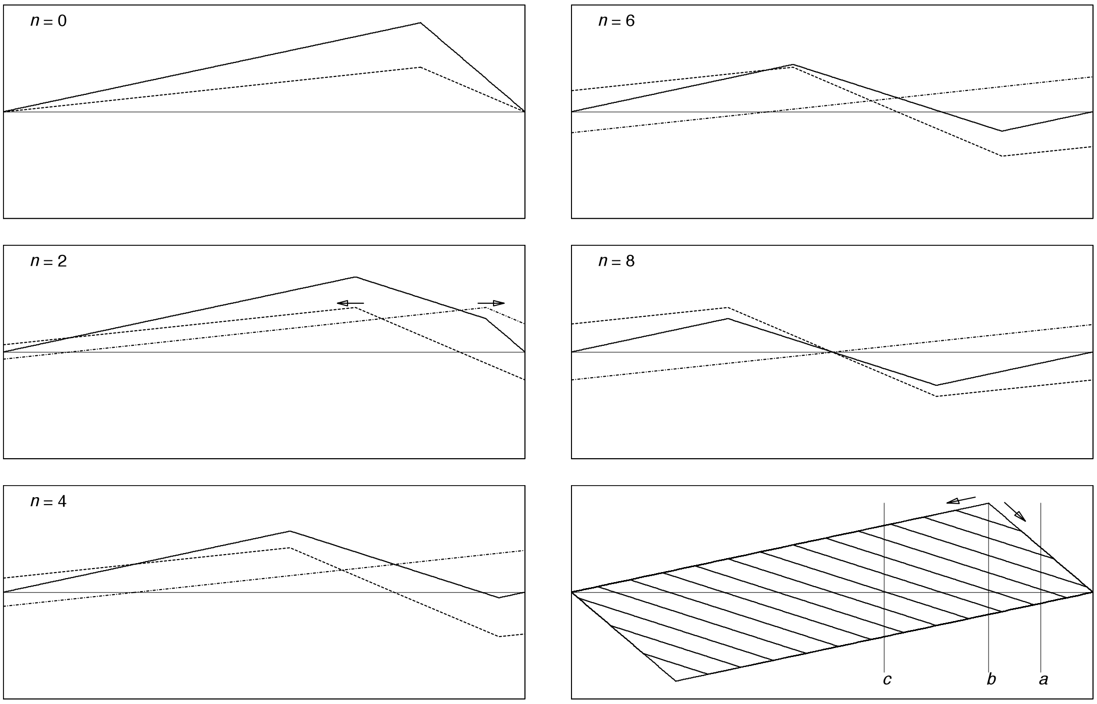

Fig. 1.2: Propagation of a triangular wave after plucking the string. The phase shift is $\varphi = n \cdot \pi / 16$.
The string (indicated as the bold line) is modeled as superposition of two partial waves running away from each other. The abscissa is the coordinate along the string (length of the string); the ordinate is the lateral displacement. A parallelogram yields the delimitation line for the string displacement (lower right). These diagrams are not time functions!

The actual string vibration is the sum of **two partial** waves running in opposite directions. Both triangularly displaced partial waves run at constant speed. The particle velocity of each point on the string is constant per section; however, the movement in one direction happens with a different particle velocity compared to the movement in the other direction. Superimposing both waves yields an unexpected result: each location on the string is either at rest, or it vibrates with the constant (!) **particle velocity** $\pm v$. String locations close to the nut or to the bridge do not vibrate more slowly but during a shorter time compared to locations at the middle of the string (**Fig. 1.3**).

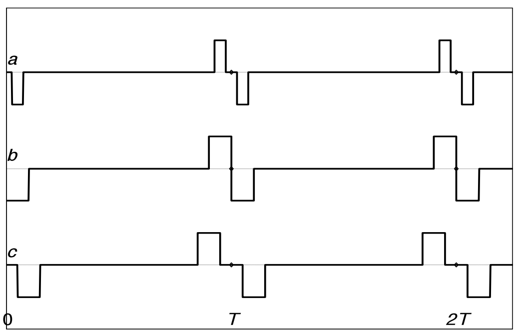

Fig. 1.3: Time function of the particle velocity of the string at three different points a, b, c (see Fig. 1.2 for comparison). From this, the time function of the displacement of these points can be derived via integration. For the v-spectrum, the superposition of two line spectra with phase-shifted si-envelope results. A temporal integration corresponds to a division by $j\omega$ in the frequency domain.

In this model-consideration it is important to distinguish the actual string oscillation (measurable in reality), and the components from which it is put together in the model. The partial waves may not be considered in isolation; they are "artificially generated" to support the visualization of the concept.

