1.3 Inharmonic partials

The simple ideal string has a length-specific mass m', and a tension-stiffness $\pi^2 \Psi/L$ created by the tension force Ψ. Conversely, the real string also includes a flexural stiffness that impedes bending the string – this is an undesirable effect that causes **dispersive wave propagation**. The heavier the string is, and the less it is tensioned, the more the flexural stiffness manifests itself (i.e. especially in the bass strings of the guitar). To achieve an improvement, heavy strings are wound with thin wire of one or more layers. The flexural stiffness is then predominantly determined by the thinner core, while a high mass loading is still possible. However, since the core cannot be made arbitrarily thin, the impact of dispersion may only be reduced but cannot be removed. Precise analyses indicate a propagation speed $c(f)$ that increases towards higher frequencies. It causes the partials to "spread out" and loose their harmonicity to a certain degree. Therefore, the term "harmonics" is incorrect in the strict sense of the word and may be replaced by the term "partial".

1.3.1 Dispersion in the frequency domain

In a linear (or at least linearized) system, any oscillation shape may be represented as a superposition of single mono-frequent oscillations. The propagation of a transversal wave is described by the **wave equation**. A position- and time-dependent transverse displacement $\underline{\xi}(z,t)$ is created along the propagation direction z, with the temporal derivative being the particle velocity.

$$\underline{\xi}(z,t) = \hat{\xi} \cdot e^{j\varphi_0} \cdot e^{j\omega t} \cdot e^{-jkz} \qquad \text{Wave equation}$$

In this equation, $\hat{\xi}$ represents the oscillation amplitude, φ_0 indicates the phase angle at the position $z = 0$ and at the point in time of $t = 0$, ω is the angular frequency, and k is the wave number. The angular frequency yields the periodicity in time $T = 2\pi/\omega$; the wave number yields the periodicity in space $\lambda = 2\pi/k$. For a fixed position z, the phase grows linearly with the time t, for a fixed point in time, the phase decreases linearly with the position z:

$$\varphi(z,t) = \varphi_0 + \omega t - kz \qquad \text{Phase function}$$

The periodicity in space (wave length λ) and the periodicity in time (oscillation period T) are linked via the propagation speed (= phase speed) c:

$$c = \omega/k = \lambda/T \qquad \text{Propagation speed}$$

A steady free oscillation can only originate if all reflections running in a z-direction superimpose with the same phase, i.e. if the phase shift across the length $2L$ amounts to an integer multiple of 2π:

$$\left.\Delta\varphi = n \cdot 2\pi = k \cdot 2L \quad \right\} \quad f_n = \frac{n \cdot c}{2L} = n \cdot f_G \qquad \text{Frequencies of partials}$$

In this equation, the propagation speed c is assumed to be frequency-*in*dependent; the partials f_n are then situated at integer multiples of the fundamental frequency.

However, in reality the string features **dispersive wave propagation** (i.e. the propagation speed is frequency dependent): high-frequency signal run at higher speeds than low-frequency signals, and therefore frequencies of the partials grow progressively (i.e. are spread out) with increasing frequency. The underlying mechanism is the already mentioned flexural stiffness that manifests itself in particular in oscillation shapes with strong curvature (i.e. at small wave-lengths = at high frequencies). It should be noted that this is a linear effect. The frequencies of the **inharmonically** spread out partials can be calculated with the following formula [appendix]:

$$f_i = n f_G \sqrt{1 + b n^2} \qquad \text{with} \qquad b = \left(\frac{\pi \kappa^2 D_A}{8 L^2 f_G} \right)^2 \cdot \frac{E}{\rho}$$ Spreading of partials

Herein, the symbols mean: f_i = frequency of inharmonic partial, f_G = fundamental frequency without dispersion, n = order of the respective partial, b = parameter of inharmonicity, E = Young's modulus (approx. $2 \cdot 10^{11}$ N/m^2), D_A = outer diameter, κ = core- / outer-diameter, L = length of the string, ρ = density.

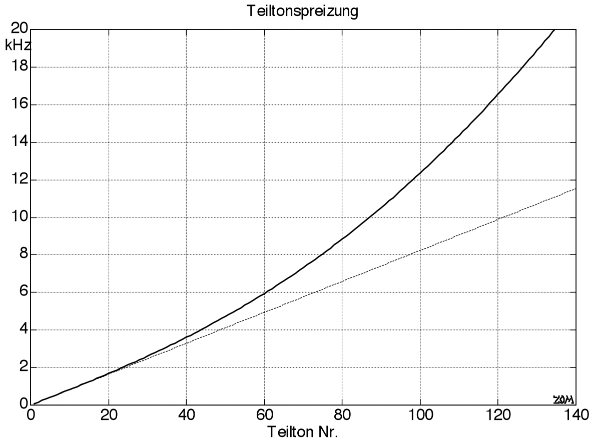
With a **solid** string of a diameter of 1,2 mm tensioned such that a fundamental frequency of 82,4 Hz results, the dispersion would detune the 20th partial from 1648 Hz to 2774 Hz – quite a considerable effect. Using, instead of a solid string, a **wound** string of the same length-specific mass, the flexural stiffness is reduced – and so is the inharmonicity. In wound guitar strings, the core diameter is smaller than the outer diameter by a factor of about 0.3 to 0.6. The density ρ will be about 7900 kg/m^3 for solid strings, while for wound strings the effective density $\overline{\rho}$ is about 10% less than the core density (Chapter 1.2). Given a wound E_2-string (with an outer diameter of 1,3 mm), the calculation yields (using b = 1/8141) a spreading of the 20th partial from 1648 Hz to 1688 Hz, i.e. by 2,5%.

In the formula of the spreading-parameter b, the length of the string L occurs with the power of 4, while the fundamental frequency only occurs with the power of 2. If, for example, fretting the octave halves the string-length, the percentile in detuning of the 20th partial increases from 2,5% to 9,5% – that is from just shy of a half-step to three half-steps. However: the 20th partial of the fretted octave lies in a different frequency range, and the direct comparison between the 20th partial of the open string and the 10th partial of the octave shows the same detuning of (2,5%). In other words: for a given string and the same absolute frequency, the inharmonicity is always of the same strength irrespective of the fretted note.

The **down-tuning** of a guitar also increases the inharmonicity: if – in the above example – the low E-string is down-tuned by a whole step (82.4 → 73.4 Hz) and the regular-tuned open E-string is compared with the down-tuned E-string fretted at the 2nd fret (i.e. in both cases we have the note E_2), the inharmonicity of the 20th partial is at 2.5% for the regular tuning, and 3.9% for the down-tuning.

At this point we shall not investigate how far these inharmonicities of the partials are actually audible; details about the topic are included in Chapter 8.2.5. [10] reports about hearing experiments, and in [2] a computation method for piano strings is developed.

Fig. 1.8 shows the relationship between the order *n* of the partial and the spread frequency f_i as it can be observed for a wound low E-string of a diameter of 1,3 mm. The fundamental frequency is 82,4 Hz, the spreading parameter is b = 1/8000.



$$f_n = n \cdot f_G$$

$$f_i = n \cdot f_G \sqrt{1 + bn^2} = f_n \sqrt{1 + bn^2}$$

$$f_n = f_G \sqrt{\frac{\sqrt{1 + 4b(f_i / f_G)^2} - 1}{2b}}$$

Fig. 1.8: Inharmonic spreading of the partials for a low E-string. The thin line marks a harmonic relation. “Teiltonspreizung” = spreading of the partials; “Teilton Nr.” = partial no.

Fig. 1.8 attributes to a given partial its spread-out frequency. For the following considerations, however, the reverse relationship is required, as well: we have a partial at a given frequency f_i, and want to know how much was it spread, or what its frequency f_n is. **Fig. 19** provides the answer. The abscissa f_i shown corresponds to the ordinate in Fig. 18.

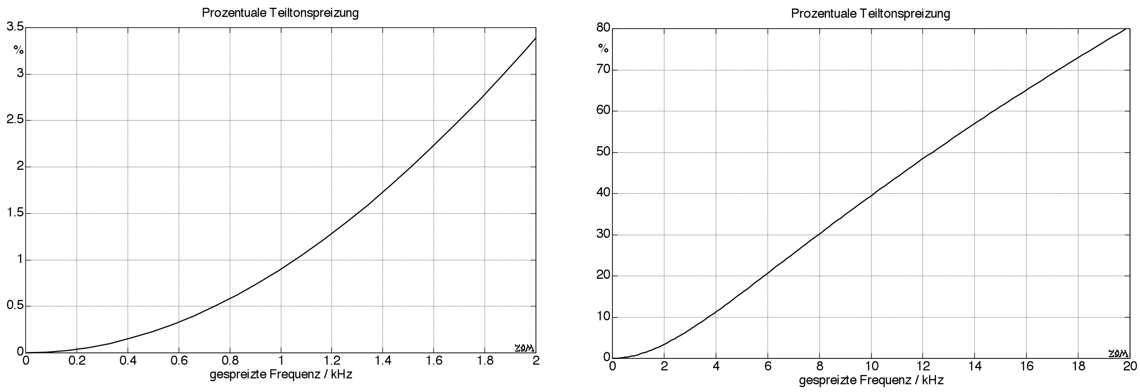


Fig. 1.9: Spreading (on a percentage basis) of partials as a function of the (spread-out) frequency (low E-string), b = 1/8000. “Prozentuale Teiltonspreizung” = spreading of partials on a percentage basis; “gespreizte Frequenz” = spread frequency.

Only at the discrete frequencies $f_i[n]$ with *n* = *integer* we find an in-phase superposition of all waves running in the same direction. For a full revolution ($z = 2L$), the phase shift amounts to $n \cdot 2\pi$, and the travel time required corresponds to the *n*-fold period *T* of the oscillation, i.e. $\tau_p = n / f_i$. Because here the travel time for a specific phase is referred to (e.g. for the zero crossing), the term used is the **phase delay** τ_p, with the corresponding propagation speed being the **phase speed** c_p.

$$\tau_p = \frac{n}{f_i} = \frac{1}{f_G \sqrt{1 + bn^2}} \qquad \text{for } z = 2L; \qquad c_p = \frac{2L}{\tau_p} = 2Lf_G \sqrt{1 + bn^2}$$

When using the formulas giving phase delay and phase speed, we need to bear in mind that the spread-out frequency is used. It is for this reason that the right-hand side of the equation should contain f_i but not f_n:

$$\tau_p(2L) = \frac{\sqrt{2}}{f_G\sqrt{1+\sqrt{1+4b\cdot(f_i/f_G)^2}}}; \qquad c_p = \sqrt{2}\cdot L\cdot f_G\cdot\sqrt{1+\sqrt{1+4b\cdot(f_i/f_G)^2}}$$

Fig. 1.10 depicts the frequency dependency of the phase delay and the phase speed. On the abscissa we find the spread frequency f_i, i.e. the frequency where the oscillation actually occurs. The calculation here is done for the low E-string (E_2) with b = 1/8000.

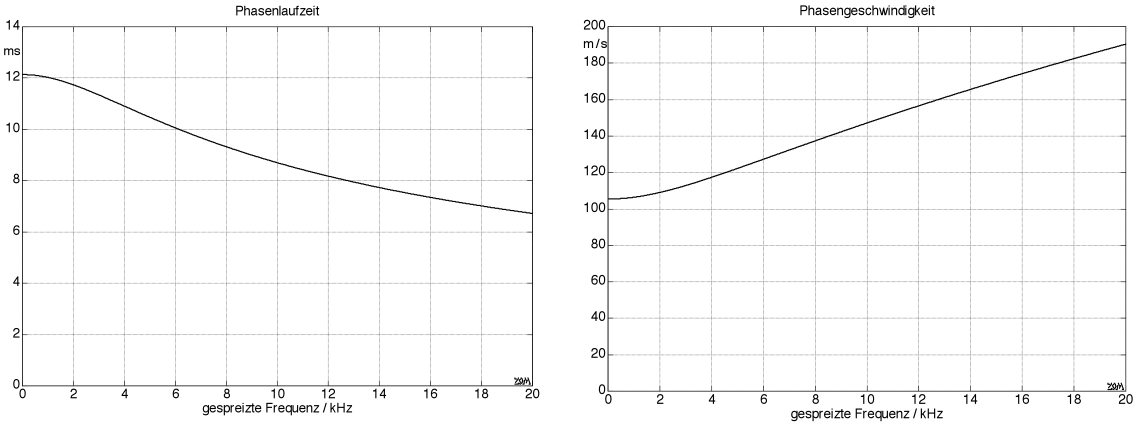


Fig. 1.10: Phase delay ("Phasenlaufzeit") (z = 2L) and phase speed ("Phasengeschwindigkeit"), low E-string, b = 1/8000. "gespreizte Frequenz" = spread-out frequency

For the following considerations, theoretical calculations are compared to measurements. An **Ovation guitar** (Viper EA-68) constitutes the measuring object – it includes a piezo-pickup mounted in the bridge. The Viper is not a typical Ovation: its body has a thickness of 5 cm, and being largely solid it can be counted as a solid-body guitar. The built-in amplifier was not used; rather, the pickup was directly connected to an external measuring amplifier featuring very high input impedance. For the majority of the measurements, D'Addario Phosphor-Bronze strings EJ26 were deployed (.011 **–** .052). If not specified otherwise, the guitar was in standard tuning E-A-D-G-B-E.

Fig. 1.11 juxtaposes calculation and measurement. There is a problem in principle with the (or any) spectral analysis: to obtain a high frequency resolution, a measurement with a long time duration is necessary – analysis-bandwidth and -duration are reciprocal, after all. However, with long measurement duration, dissipation makes itself felt at high frequencies – the signal is not in steady-state anymore. Any measurement will therefore represent a compromise. In Fig. 1.11, the duration of the analysis amounts to 85 ms, and instead of narrow spectral lines the result are funnel-shaped extensions (DFT-leakage). Pointing upwards, the tips of the funnels indicate the frequency of the respective partial; the minima of the curves are of no significance. To compare, Fig. 1.11a holds (as dots) the calculation results for *harmonic* partials: the correspondence is weak – at 2,3 kHz, the frequency-discrepancy is already as big as the distance between two partials. Fig. 1.11.b shows the *spread* partial frequencies with a significantly better correspondence. Any remaining differences will be discussed later – as will be the frequency-dependence of the level.

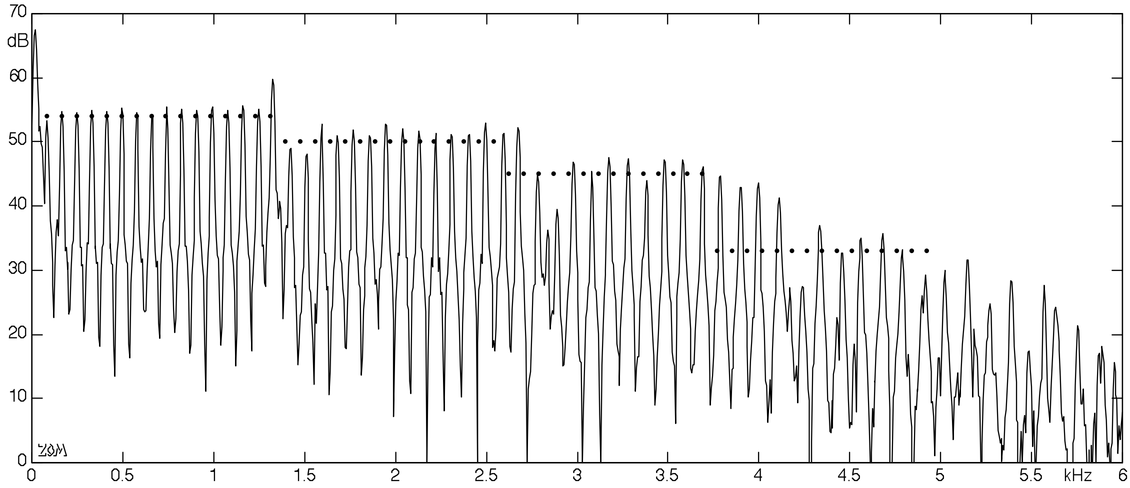


Fig. 1.11.a: Measured spectrum (lines), calculated *harmonic* partials (dots).

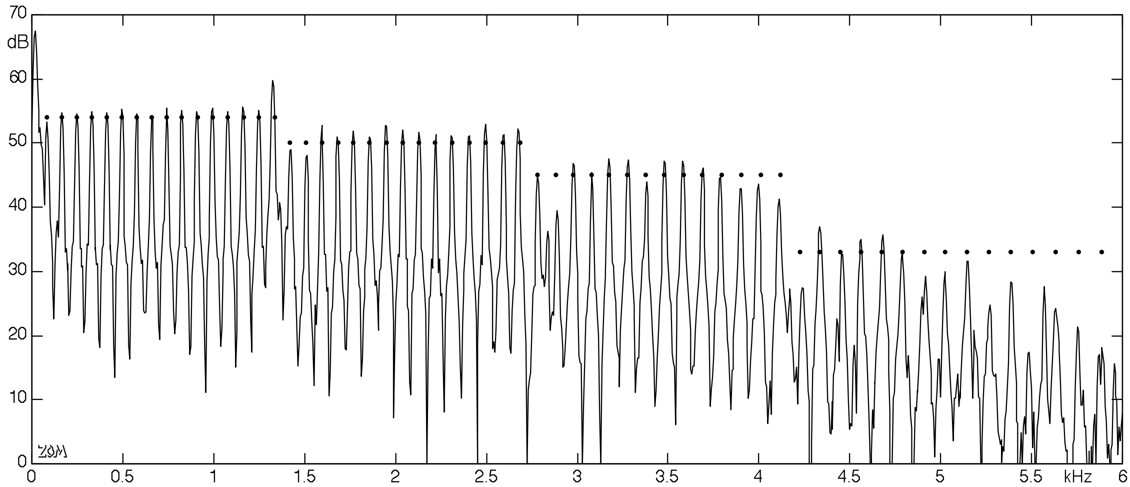


Fig. 1.11.b: Measured spectrum (lines), calculated *spread* partials (dots, b = 1/8500).

The problem mentioned above regarding the selectivity occurs particularly in spectrograms. To generate them, many single spectra are superimposed as color- or grey-scale-coded lines (**Fig. 1.12**). Herein, the level (dB-value) is entered as a function of time (ordinate) and frequency (abscissa). However, a spectrum can never be determined at a *point* in time but only for a time-interval. If we shorten the duration corresponding to this interval in order to obtain a good time–selectivity, the spectral selectivity deteriorates. In Fig. 1.12, the time window has an effective length of 1,9 ms, with the effective bandwidth being 526 Hz. In the low-frequency range, red/yellow bars follow each other with an interval of 12 ms; these are the reflections of the plucking process. The reciprocal of this periodicity corresponds to the fundamental frequency. Towards higher frequencies, the intervals become shorter – corresponding to the spreading of the frequencies of the partials. The quantitative evaluation is not (yet) a good match for Fig. 1.10: as is evident, the inharmonicity occurring towards higher frequencies is much more pronounced on Fig. 1.12.

The reason for these apparent discrepancies is found in the way the analysis is done: a spectrogram shows the envelope shapes corresponding to given frequency ranges, and not the propagation of a certain oscillation phase. For this reason, it is the **group delay** that needs to be considered for the comparison, and not the phase delay. The phase delay is the negative quotient of phase and angular frequency, while the group delay is the negative differential quotient.

$$\tau_p = -\varphi/\omega \qquad \text{Phase delay}$$

$$\tau_g = -d\varphi/d\omega \qquad \text{Group delay}$$

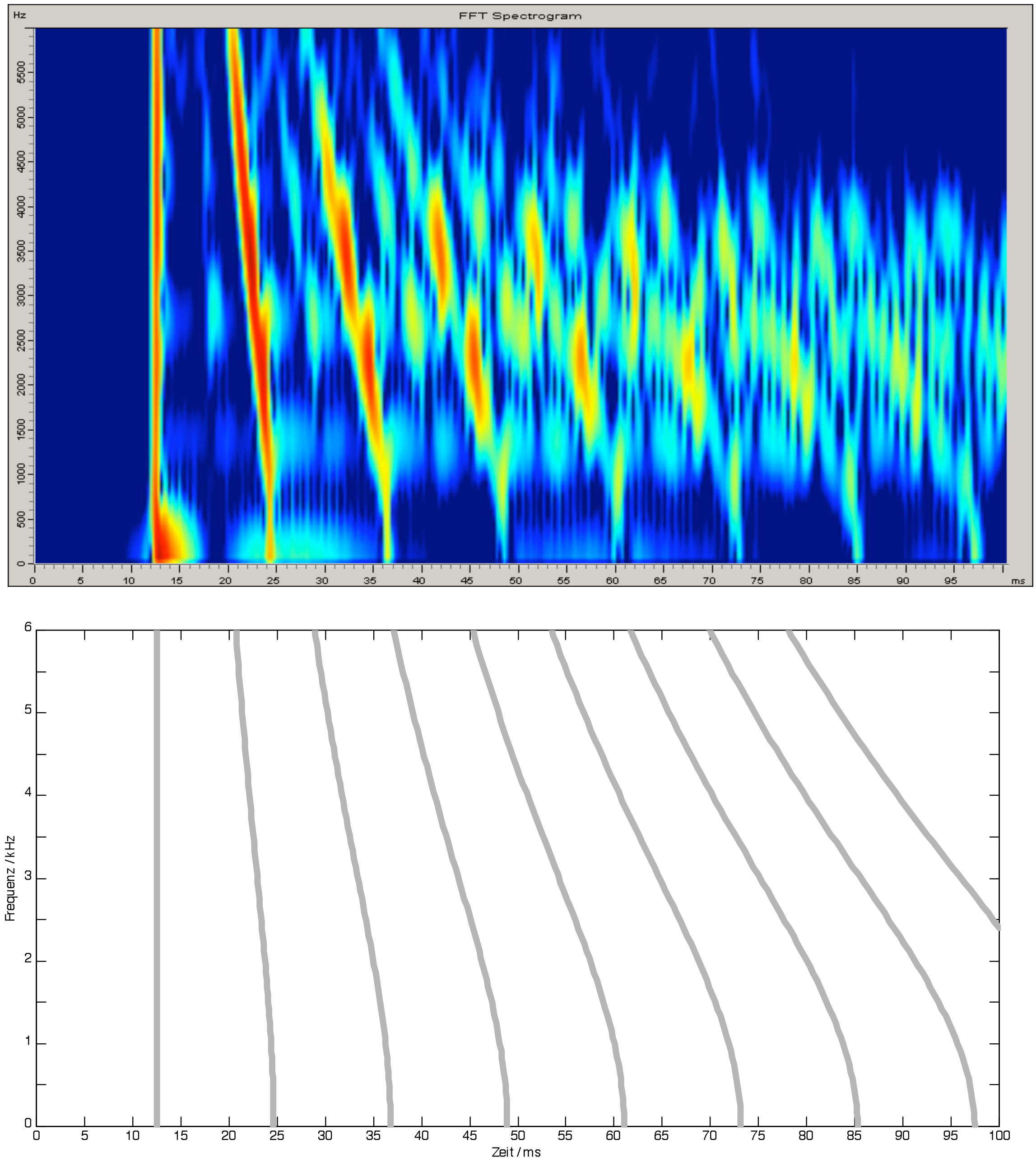


Fig. 1.12: Spectrogram of the plucking process of a low E-string (top); computer simulation (bottom). The resonances occurring at multiples of 1,4 kHz are due to expansion waves (Chapter 1.4). “Frequenz” = frequency.

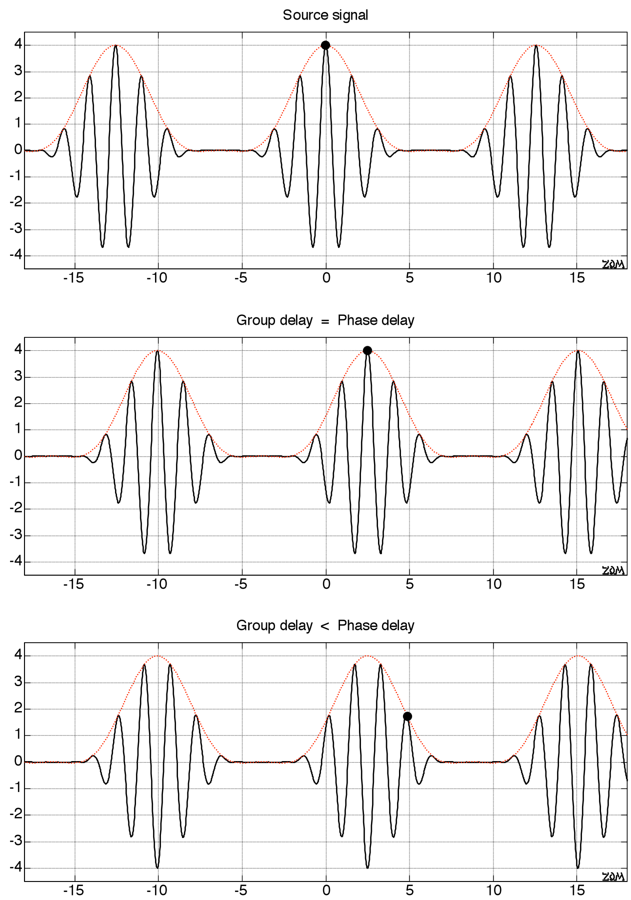


Fig. 1.13 illustrates the differences. The uppermost graph is the time function of a signal resulting from 5 neighboring tones. As this signal runs through a system with frequency-proportional phase, the envelope and the carrier below it are shifted by the same delay (middle graph). Phase delay and group delay are equal in this case.

If there is a linear, but offset relationship, phase- and group-delay differ (lower graph). The envelope is not shifted as much as a certain carrier-phase (marked here with a dot). The time function is not only shifted but has also changed its shape. Due to the frequency-independent group delay, the shape of the envelope has, however, not changed.

Fig. 1.13: Explaining the difference between phase-delay and group-delay.

For a dispersive string, the group delay for a full oscillation period is calculated as:

$$\tau_g = \frac{1}{f_G}\sqrt{\frac{1+\sqrt{1+4b(f_i/f_G)^2}}{2+8b(f_i/f_G)^2}} \qquad \text{for } z = 2L; \qquad \text{Group delay}$$

Inserting into this equation a low value for f_i (e.g. f_G) yields a group delay that is – with good approximation – the reciprocal of the fundamental frequency, i.e. about 12 ms. For higher frequencies this value drops to about 7,8 ms which is a good match to the high-frequency impulse distances observed in Fig. 1.12.

The lower section of **Fig. 1.12**, shows a computer simulation for the spectrogram depicted above it. While the differences are not to be ignored (multiple decay processes of the excited resonances and superimposed expansion waves make for an early unraveling of the original line structure), we can still see already in this simple analysis a good correspondence of the dispersive effects, From the point of view of systems theory, the dispersive propagation may be described as an **all-pass**: a linear, loss-free filter with a frequency dependent delay-time. Compared to an ideal all-pass, the vibration energy of a real string decays – but let's postpone dealing with this effect a bit. Linear filters are described by their complex **transfer function** in the frequency domain, and in the time domain by their impulse response. The **magnitude** of the transfer function of an all-pass is equal to one for all frequencies (loss-free transmission). If the **phase** of the all-pass transfer function were zero, input and output signal would correspond (trivial case). If the phase were proportional to the frequency, all frequency components would be delayed by the same delay time, and the system would not be termed all-pass, but delay line. In a non-trivial all-pass, the phase $\varphi(\omega)$ is not proportional to the frequency. The phase delay thus is frequency-dependent – for a string this occurs in such a way that high frequencies appear at the output of the all-pass after a shorter delay than low frequencies.

Of course, the delay time also depends on the distance traveled. Assuming precise manufacture with place-*in*dependent mass and stiffness along the string, the string represents a **homogenous transmission line**: the propagation speed is frequency-dependent but place-independent. The phase shift thus shows proportionality to the traveled distance – at any frequency (with a frequency-dependent proportionality factor). This assumption corresponds well with the real string; we find somewhat more serious problems with the places of reflection at the nut and bridge … we will have to look into this more specifically later.

It already has been explained with respect to Fig. 1.12 that for the propagation of envelopes it is not the phase delay that is important, but the group delay. In non-dispersive systems, phase delay and group delay are identical, but in the dispersive string the group delay is smaller than the phase delay. As a description of the transfer characteristics of an all-pass, we typically find the **frequency response of the group delay** in the frequency domain, and the **impulse response** in the time domain; both characteristics are equivalent and can be converted one into the other.

The frequency responses of phase delay and group delay are shown in **Fig. 1.14**. The abscissa is the spread-out frequency f_i, rather than the n-fold fundamental frequency.

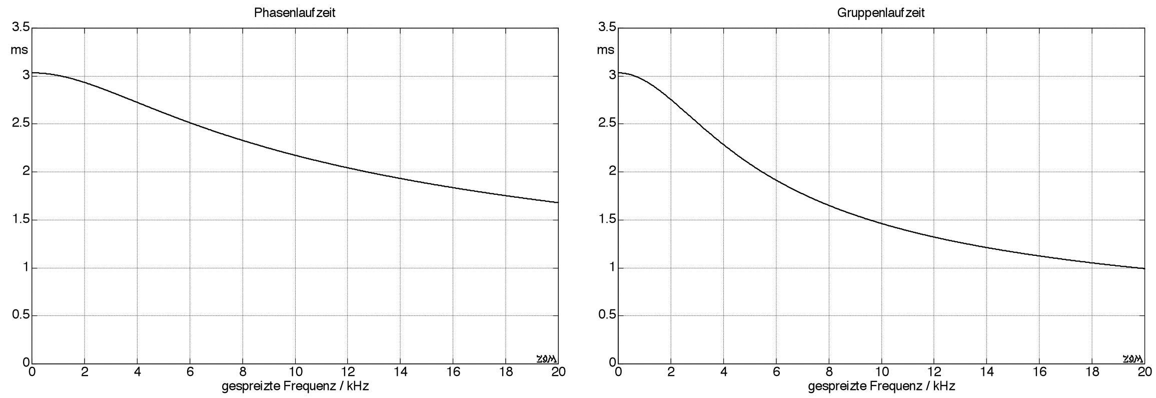


Fig. 1.14: Phase delay and group delay across half the string length (from bridge to mid-string), E_2, b = 1/8000. "Phasenlaufzeit" = phase delay; "Gruppenlaufzeit" = group delay; "gespreizte Frequenz" = spread frequency.