1.3.2 Dispersion in the time domain

Guitar strings are plucked with the finger or a plectrum (pick). A slowly increasing force pulls the string from its rest position, then this force suddenly stops, and the string executes a free damped oscillation. The idealized time function of this excitation is a **force-step**: at the point in time $t = 0$ the force jumps from an initial value to zero. Starting from the plucking point, a step-wave travels in both directions. However, this wave will now change its shape due to the dispersion: the high-frequency components of the step travel faster than the low-frequency ones. The step is being pulled apart in both the frequency- and time-domains. From the viewpoint of systems theory, the dispersive propagation may be modeled by an **all-pass**. The latter is a linear, loss-free filter with frequency-independent transfer coefficient and frequency-dependent delay time (Chapter 1.3.1). Transfer function and **impulse response** represent the transmission-relevant quantities of an all-pass.

The impulse response of a linear system is formed by the inverse Fourier-transform of its transfer function. Convolution of any arbitrary input signal with the impulse response yields the output signal. According to this definition, if the system is stimulated at its input e.g. with a step, the output signal is the result of a convolution of step and impulse response. For this special case, a simplification is possible: the step is the (particular) temporal integral of the impulse. Like differentiation, integration is a linear operation, and therefore the sequence of impulse/integrator/system may be exchanged for impulse/system/integrator (commutative law). The step-response of a linear system therefore corresponds to the integrated impulse response, just like the impulse response corresponds to the derivative of the step response.

The model system used in the following to emulate the plucked string is an all-pass with a step-function being fed to its input.

In **Fig. 1.15.a** we see on the left the measurement result from an E_2-string plucked halfway between nut and bridge ($z = L/2$). On the right, the step-response of an all-pass is shown for comparison. There are clear differences but also some commonalities: the step response permanently switches its polarity after 3 ms; this delay time corresponds to the low-frequency group delay for half the string length. From about 1 ms – corresponding to the shorter high-frequency group delay – we see fast oscillations. In the output signal of the piezo, the high-frequency oscillations have more damping (treble cut). Moreover, there is a dip at 0 – 2 ms caused by the plectrum. After 3 ms, decay processes of the longitudinal resonances appear (Chapter 1.4) – these are not present in the simulation of the all-pass.

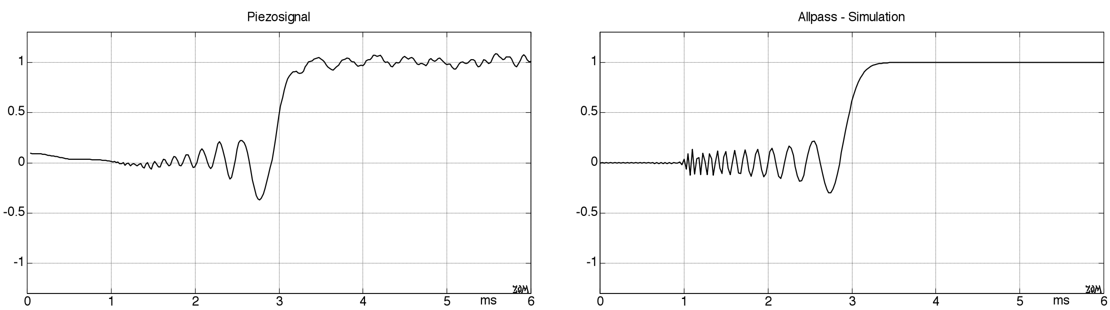

Fig. 1.15.a: Piezo-signal (left) and simple simulation of an all-pass (right); excitation by a step at mid-string and $t = 0$. For the piezo-signal, sign and offset were chosen for best fit.

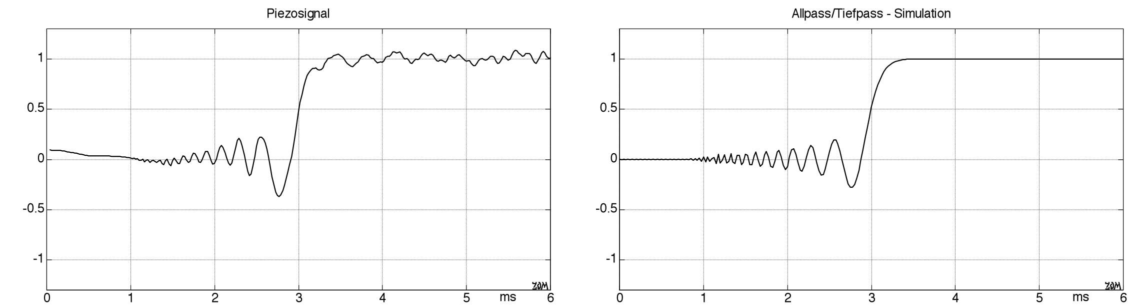

Fig. 1.15b: Piezo signal (left) and all-pass/low-pass simulation; step-excitation at mid-string and $t = 0$. ”Tiefpass” = low-pass.

For **Fig. 1.15.b**, the same all-pass as in Fig. 1.15a was used but supplemented by a simple low-pass in order to model the treble-cut (dissipation). The amplitude of the early oscillations can effectively be damped this way.

Two remarks regarding the **bandwidth**: the piezo-signal was sampled with 48 kHz. It received a band-limitation at 20 kHz by a low-pass filter, just like the all-pass simulation. The lower frequency limit of the measuring amplifier is 2 Hz. DC-coupling is not purposeful and would only crate offset-problems. As a consequence, the zero-point of the ordinate is arbitrary. Moreover, the sign was reversed such that the step happens from zero to positive values as is customary in systems theory.

Fig. 1.16 depicts a longer section taken from the piezo signal. With increasing time, the step is pulled more and more apart, and therefore no “period” is equal to another. Assuming, for one revolution ($z = 2L$), 12 ms at low frequencies and 4 ms for higher frequencies, the step is spread out already across several “periods” after 5 revolutions with 60 ms | 20 ms. A short-term spectrum measured over a short time duration therefore captures signal components that have been reflected differing numbers of times, depending on the frequency range.

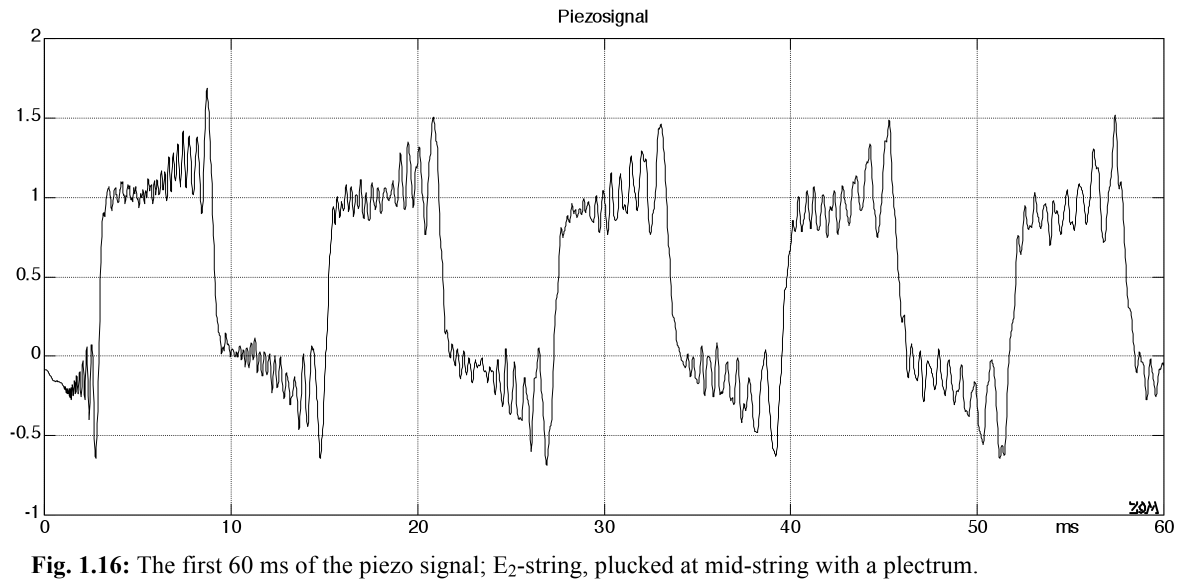

Fig. 1.16: The first 60 ms of the piezo signal; E_2-string, plucked at mid-string with a plectrum.

