1.6 The decay process

After being plucked, the sting vibrates in a free, damped oscillation process. “Free” implies that no further energy is injected; “damped” indicates that vibration energy is converted into sound and caloric energy (radiation, dissipation). Any further string damping (e.g. via the fingertips of the palm of the hand) shall not be considered here at this time.

1.6.1 One single degree of freedom (plane polarization)

The simplest oscillation system consists of a mass, a spring, and a damper. The mass force is proportional to the acceleration (inertia, NEWTON), the spring force is proportional to the displacement (stiffness, HOOKE), and the damper force is proportional to the (particle) velocity (friction, STOKES). The time derivative of the displacement yields the velocity; the time derivative of the velocity yields the acceleration [3].

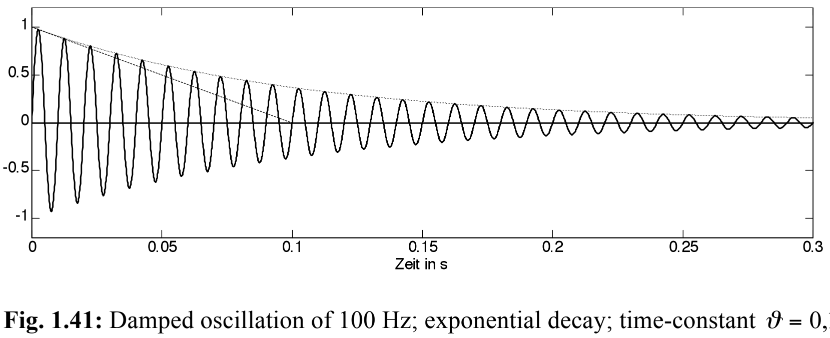
After the excitation a “periodic” oscillation of the frequency f_d results. Instants of equal phase (e.g. maxima, zeroes, and minima) occur at equal distances in time – which led to the us of the term **period** $T = 1/f_d$. However, signal theory does not actually see this decay process as a periodic signal: due to the exponential decay, the individual periods fail to be identical. Mechanics, on the other had, do use the term periodic vibration here because the duration of the periods in time-invariant (... non est disputandum).

The resulting envelope has three parameters: the frequency f_d, the initial phase φ, and the time constant of the envelope ϑ. In this general form, the equation for the oscillation is:

$$\xi(t) = \hat{\xi} \cdot e^{-t/\vartheta} \cdot \sin(2\pi f_d t + \varphi), \qquad t \geq 0 \qquad \text{Oscillation equation}$$

For $t = 0$, the e-function yields 1; with increasing time, it decreases towards 0. The phase shift φ may be taken to be zero for the first considerations. The time constant ϑ determines how fast the oscillation decays: the smaller ϑ is, the faster the decay. Instead of ϑ, literature offers a multitude of other parameters, as well – they can easily be converted into each other. The letter τ is frequently used for the time-constant; in the present context we will rely on this letter only when we get to the calculation of levels. What needs to be avoided in particular is confusion between the degree of damping and the decay-coefficient, since the latter is sometimes also designated with ϑ!

It may be the displacement, the (particle) velocity, or the acceleration that represents the physical oscillation. A sensor converts these quantities into a voltage $u(t)$ that subsequently is analyzed.



$f_d = 100\,\text{Hz}$

$\vartheta = 0{,}1\,\text{s}$

Fig. 1.41: Damped oscillation of 100 Hz; exponential decay; time-constant $\vartheta = 0{,}1\,\text{s}$.

Given mass *m*, spring-stiffness *s*, and friction *W*, we calculate frequency and time-constant:

$$f_d = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \frac{1}{\vartheta^2}} \qquad \vartheta = \frac{2m}{W}$$ Parameters of the oscillation

If the friction *W* is set to zero, the un-damped system results. It has an infinite time-constant: the *e*-function now has the constant value 1, and the vibration does not decay anymore. A weakly damped vibration with a frequency $f_d = 100$ is shown in **Fig. 1.41**. The shape of the *e*-function is indicated as a dashed line with its tangent crossing zero at ϑ. At the point in time of $t = \vartheta$, the envelope has decreased from 1 to $1/e \approx 0{,}37$.

In instrumentation, the decay process is often depicted as **level-curve**. Level is a logarithmic measure that may be determined in various ways. It always constitutes a time-average over a weighted measurement interval; the averaging is done using the squared signal quantity. We often see an exponential averaging where the weighting is of exponential form, and is done such that the signal components lying further back in the past contribute less prominently to the measurement. The **averaging time constant τ** is specified as parameter of the exponential averaging; the value $\tau = 125\,\text{ms}$ is used frequently, with the corresponding standardized way of averaging being labeled **FAST**. The decay constant ϑ of the dampened oscillation must not be confused with the averaging time constant τ of the level measurement.

The level measurement comprises three consecutive operations: squaring, averaging, and logarithmizing. Squaring and logarithmizing are non-linear operations; the order of sequence must therefore not be interchanged. It is only the averaging that is a linear filter operation: a 1^{st}-order low-pass in the case of the level measurement. In the time domain, the averaging is described by a convolution [6]: the result of the averaging corresponds to the convolution of squared signal and impulse response *h*(*t*) of the averager. For damped oscillations we get:

$$m(t) = h(t) * u^2(t) = \int_0^t \left(\frac{1}{\tau} e^{\frac{\psi - t}{\tau}}\right) \cdot \left(\hat{u} \cdot e^{-\psi/\vartheta} \cdot \sin(\omega_d \psi)\right)^2 \cdot d\psi$$ (for causal signals)

$$h(t) = \frac{1}{\tau} e^{-t/\tau} \qquad u(t) = \hat{u} \cdot e^{-t/\vartheta} \cdot \sin(\omega_d t) \qquad \omega_d = 2\pi f_d$$

Here, *h*(*t*) is the impulse response of the averager, *u*(*t*) is the damped oscillation, the star symbol stands for the convolution. The average *m*(*t*) is calculated for the point in time *t* with the time-variable ψ integrated from 0 to *t*. Therefore, the **average value** *m*(*t*) does in this case not indicate the average over the whole decaying oscillation but the average from the excitation to the (variable) point in time *t*. The averaging time constant τ is large compared to the oscillation period *T*; the contribution of the sine function can thus be disregarded in good approximation. Using this, the time-variant average is:

$$m(t) = \frac{\tilde{u}^2}{1 - 2\tau/\vartheta} \cdot \left(e^{\frac{-2t}{\vartheta}} - e^{\frac{-t}{\tau}}\right) \qquad \tilde{u} = \hat{u}/\sqrt{2} \qquad \text{for } 2\tau \neq \vartheta$$

When calculating levels we need to consider that we are working with a squared signal, which is why we need to opt for the formula for power levels. The reference value needs to be chosen such that the correct absolute value results for the steady case ($\vartheta \to \infty$). Using, on the other hand, $u_0 = \tilde{u}$, we get the relative level that decays starting from 0 dB.

$$L(t) = 10\lg\left(m(t)/u_0^2\right)\text{dB} \qquad \text{dB = decibel} \qquad u_0 = \text{reference value}$$

Fig. 1.42 shows the course of the level of a damped oscillation determined via exponential averaging. The time-constant of the damping is $\vartheta = 4\,\text{s}$. Having an understanding of the equation of the oscillation, we could also give the exact course of the level. To do that, it is merely necessary to logarithmize the *e*-function (shown as a dashed line). The level determined via measurements deviates significantly from this calculation. In the figure, we see two graphs with the averaging time-constants 0,125 s and 0,5 s, as well as the theoretical behavior (dashed).

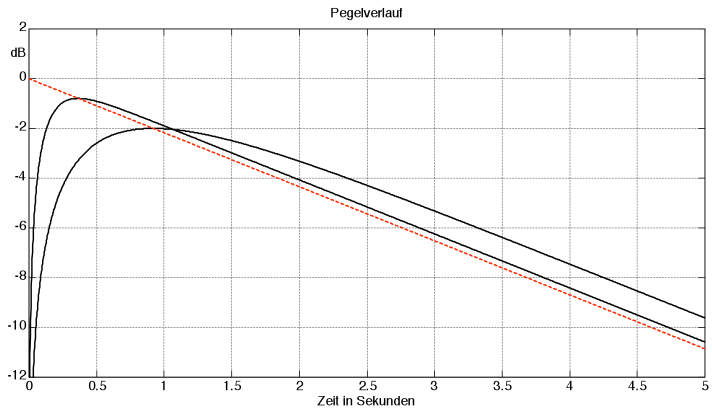


Fig. 1.42: Level of an exponentially damped oscillation. Damping time-constant ϑ = 4 s, averaging time constanr, τ = 125 ms and 500 ms. For 500 ms, the asymptote is too high by 1,2 dB, and for 125 ms, it is too high by 0,3 dB.
"Pegelverlauf" = course of the level; "Zeit in Sekunden" = time in seconds

After a short attack phase (mainly determined by τ), the level drops off with approximately the time constant ϑ. As is evident, the measurement curves run in parallel to the exact values after a short time, but remain too high. Therefore the slope – and thus the system damping – can be determined with good accuracy; for measurements of absolute values, however, considerable errors may arise. Using $L(t)$, the level difference is calculated as:

$$\Delta L = 10\lg\frac{1}{1-2\tau/\vartheta}\text{dB} \qquad\qquad \vartheta = 10\tau \;\Big\}\; \Delta L \approx 1\text{dB}$$

The shorter the averaging time-constant gets relative to the damping time-constant, the more exact the tracing of levels via measurements becomes. Still, the averaging time-constant must not be chosen too short, either, because then the (squared) oscillation may not be fully averaged anymore, and ripples in the level-graphs would result.

Moreover, Fig. 1.42 indicates that the measured **level maximum** is lower than expected. The position of the maximum is determined via differentiating and zeroing:

$$t_{\max} = \frac{\ln(2\tau/\vartheta)}{2/\tau - 1/\tau} \qquad\qquad m_{\max} = \tilde{u}^2 \cdot \left(\frac{\vartheta}{2\tau}\right)^{\frac{1}{1-\vartheta/2\tau}}$$

The larger the averaging time-constant is chosen, the lower the maximum.

From a signal-theory point-of-view, a damped oscillation belongs with **energy signals**. The signal energy is derived as integral over the squared signal value; it differs from the physical energy:

$$E_{Signal} = \int_{-\infty}^{\infty} u^2(t)\,dt \qquad E_{phys} = \int_{-\infty}^{\infty} F(t)v(t)dt = \int_{-\infty}^{\infty} v^2(t)\cdot Z\cdot dt \qquad Z = \text{impedance}$$

The signal energy of the damped oscillation may be calculated from the equation of the oscillation using integration:

$$E = \int_0^{\infty}\left(\hat{u}\cdot e^{-t/\vartheta}\cdot\sin(2\pi f_d t)\right)^2 dt \quad \xrightarrow{\vartheta\cdot f_d \,>>\, 1} \quad E = \hat{u}^2\cdot\vartheta/4$$

The average value across $m(t)$ yields the same signal energy irrespective of τ. If the energy is derived via m_{max}, however, a correction is required due to $m_{max} < \tilde{u}^2$.

Besides the exponential averaging there are also other ways to average: block-averaging is done with constant weighting across a fixed time interval, Hanning-averaging uses a sine-shaped weighting. Block averaging is also called **linear averaging**, a rather confusing term that is common in the area of spectral analysis, though. While the exponential averaging is always run from the start of the signal to the point in time of the measurement (marked with a star on Fig. **1.43**), linear averaging is done from the start of the signal over an interval of fixed duration (1 s in the figure). In exponential averaging, only the end of the interval is shifted, in linear averaging, however, this is done to both start and end. **The Hanning-averaging** uses a fixed duration of the averaging (2 s in the figure), as well, but weighs the signal with a $\sin^2$. Hanning-averaging is often deployed in DFT-analyzers – as are many other DFT-windows (Blackman Kaiser, Bessel Gauß, Flat-Top, etc.).

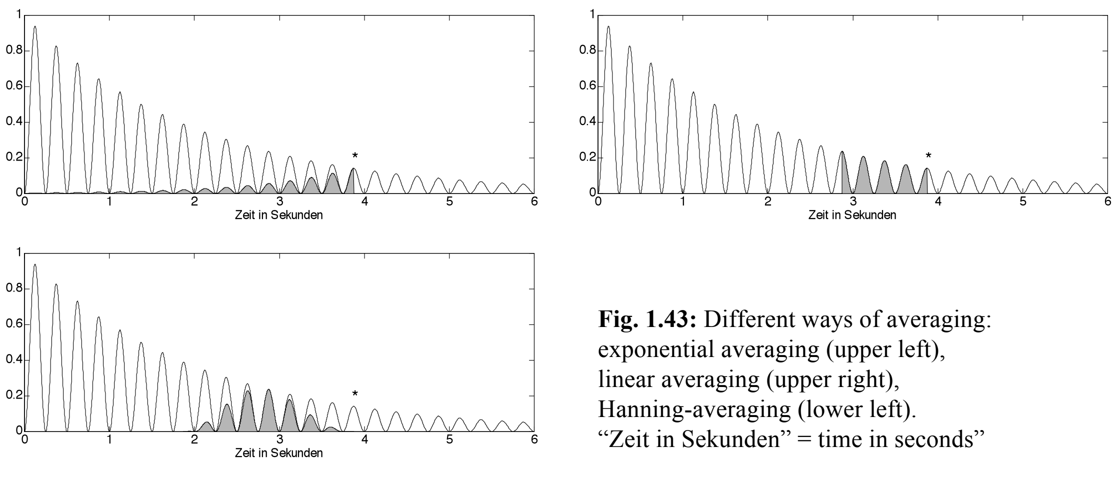


Fig. 1.43: Different ways of averaging: exponential averaging (upper left), linear averaging (upper right), Hanning-averaging (lower left). "Zeit in Sekunden" = time in seconds"

All ways of averaging are calibrated such that for steady signals (constant level), equal results are obtained. With levels varying over time, differences occur. In frequency-selective analyses (DFT, 1/3rd-octave, etc.), also further system-immanent errors contribute: a filter will react more sluggishly to the input signal as the filter band becomes narrower. In broadband level-measurements (e.g. 10 Hz – 20 kHz), no significant errors will occur, but in selective measurements of partials (e.g. 2500 Hz – 2519 Hz), they might creep in, depending on circumstances.

 Translation by Tilmann Zwicker