1.6.2 Spatial string vibrations

After a guitar string is plucked, spatial vibrations will propagate on it. The transversal waves introduced in Chapter 1.1 are of particular significance. Given that the axis along the string is taken as z-coordinate, transversal waves can propagate both in the xz-plane and the yz-plane; superpositions are possible, as well. For electric guitars, the vibration plane perpendicular to the guitar top is especially important, while for acoustic guitars the vibration parallel to the guitar top also has effects.

The wave equation includes a dependency both on place and time. However, investigations into the vibrations of guitar strings are mostly based on a fixed location (the place of e.g. pickup, or bridge) so that merely the time remains as variable. As a simplification, the string vibration occurring at a given location tends to be seen as superposition of many exponentially decaying partials (Chapter 1.6.3). In this scenario we need to consider, though, that for each partial, vibrations may appear in two planes. Sometimes one of the two vibrations has next to no effect and may be disregarded, but in some cases both need to be taken into consideration.

The following approaches first start from the assumption that plucking the string will result in two *same-frequency* vibrations orthogonal in space. The time constants of the damping ϑ are still different for the two vibrations, the effect on the output is different, and they may be phase-shifted relative to each other. At the output, both are superimposed:

$$u(t) = \hat{u} \cdot \left(e^{-t/\vartheta_1} \cdot \sin(\omega t) + d \cdot e^{-t/\vartheta_2} \cdot \sin(\omega t + \varphi) \right) \qquad d = \text{top-parallel part}$$

Particularly in acoustic guitars, the top-normal vibration is tightly coupled to the resulting sound field, and therefore vibration energy is relatively quickly withdrawn, and the damping time-constant is short. The top-parallel vibration does not lead to as efficient a radiation (d is smaller); it thus has a longer time-constant. In the level-analysis, the decay shows up with a characteristic kink (**Fig. 1.44**).

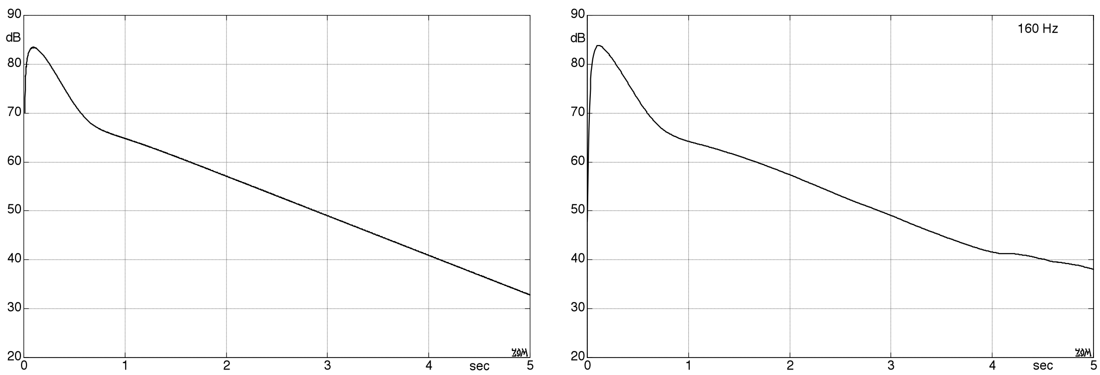

Fig. 1.44: Open E_2-string, FAST-level of the 2^{nd} partial; *left*: calculation; *right*: measurement (Martin D45V).

To confirm our hypotheses about the vibrations, two experiments were carried out. In order to adjust the neck, the OVATION Adamas SMT allows for the removal of a cover plate (of ∅13cm) in the guitar body. This detunes the Helmholtz resonance and thus changes the low-frequency coupling to the sound field. With the cover taken off, the low frequencies receive weaker radiation; the time constant should therefore be longer.

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Fig. 1.45: Left: Ovation Adamas SMT, level of fundamental (F#$_2$), with closed ("Deckel geschlossen") and removed ("Deckel geöffnet") cover plate.
Right: Ovation Viper EA-68, level of fundamental (F#$_2$), with ("mit Magnetfeld") and without influence of the magnetic field.

Fig. 1.45 (left) depicts the decay curves for the fundamental of the tone F# fretted at the 2nd fret on the low E-string. The measurements confirm the assumption. In a second experiment, a **permanent magnet** was brought close to the low E-string on an OVATION Viper. Due to the attraction force the stiffness of the string is reduced in *one* plane of vibration – the vibration frequency is thus reduced in that plane. This leads to a **beating** of the orthogonal fundamentals now slightly detuned relative to each other (**Fig. 1.45**, right).

However, even without any magnetic field, the top-normal vibration of a particular partial does not necessarily occur at the exact same frequency as that of the top-parallel vibration of the same partial. This is due to the reflection factors of the string clamping (nut, bridge) – the former are dependent on the vibration direction. The spring-stiffnesses at the edges may be different for the two directions of the vibration, resulting in slight differences in the vibration frequencies. The decay process will then include beatings that render the sound more "lively". **Fig. 1.46** shows results of calculations and, for comparison, sound pressure levels measured with an acoustic guitar (MARTIN D45V, anechoic room, microphone at 1 m distance ahead of the guitar). Various patterns emerge:

The level differences between the two sub-vibrations determine the *strength* of the interference. At a difference of 20 dB, the amplitude fluctuates merely by 10%, while at 6 dB difference the fluctuations grow to 50%. Differences in the damping determine for which *period* the beating persists. If both sub-vibrations decay with the same damping, the level-difference does not change, and neither does the beat-intensity. Conversely, if the decay is different, the beats are strongest at the instant when both levels are equal. The frequency difference determines the *periodicity in the envelope*: the larger this difference, the faster the fluctuations. Moreover, the *phase* of the sub-vibrations is of significance – in particular if different damping occurs i.e. if the beats are limited to a short time-interval. The interference-caused cancellation will only present itself if both sub-vibrations are in opposite phase during said time-interval.

Another degree of freedom comes into play if we allow for **non-linearities**. For example, the friction may depend on a higher order of particle velocity, or the spring-stiffness may depend on the displacement. This may cause, for example, that the level of a mono-frequent vibration does not decay linearly with time but shows a curvature. Addressing such aspects requires considerable effort – no corresponding investigations were carried out in the present framework.

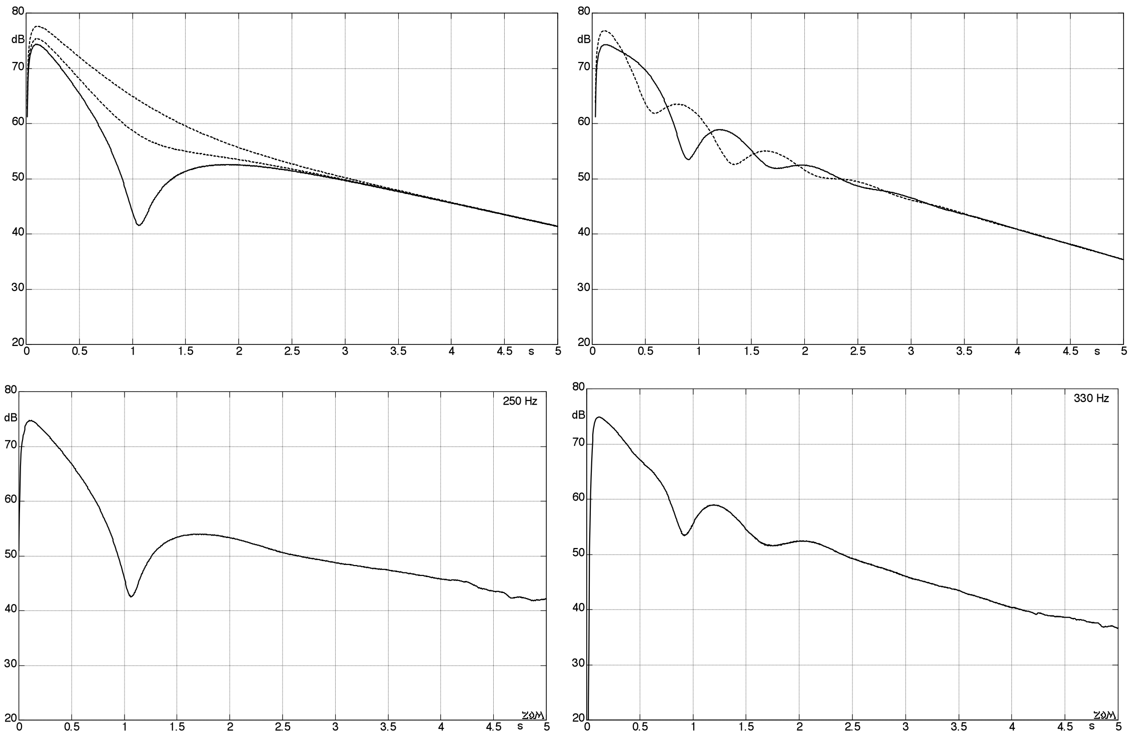

Fig. 1.46 Top: Decay processes given phase-differences. Left: both vibrations with the same frequency; right: beats due to a frequency differencde of 1.2 Hz. The damping cannot be determined precisely anymore from the initial slope of the curve. **Bottom**: measurements with a MARTIN D45V

An interesting set of curves emerges if the excitation energy remains constant while the string damping varies. First, however, we need to define more precisely the term "damping": any real string executes a damped vibration. In this case, **damping** means that vibration energy is continuously withdrawn from the string, with displacement amplitude (potential energy) and velocity amplitude (kinetic energy) decreasing over the course of time. Springs and masses store energy while resistances "remove" energy. Sure, energy cannot actually be removed – rather its mechanic incarnations are converted into caloric energy (heat); but in any case the "removed" energy is not available anymore to the vibration of the string.

In the **acoustic guitar**, we need to distinguish between the 'good' and the 'bad' losses. If all of the energy in the string is converted to sound-energy with an efficiency of 100%, we do have damping (a loss), but the objective of generating sound has been achieved with the utmost efficiency. If, conversely, 90% of the energy in the strings is converted directly into heat due to inner friction, and only 10% are radiated, we have an undesirable loss. To illustrate this with an EXAMPLE: a watering can supplies water to a flowerpot. If the water flows through a small cross-section, it will take a long time until the can is empty. With a larger cross-section, the process will be quicker – but it's always the whole of the water that arrived in the flowerpot. This situation changes if there is a hole in the bottom of the can – an additional degree of freedom is now present that influences the efficiency ◊. Applying this to the string: via tight coupling between string and sound field, the energy flows from the string quickly – the string is damped strongly but all energy reaches the sound field (100% efficiency). The efficiency drops only as friction-resistance is included in the guitar.

In **electric guitars**, the objective is entirely different. They do not need to radiate sound energy – that's taken care of by the loudspeaker. Due to the lack of radiation loss, the string damping is lower, the decay is longer – the guitar has longer/better **sustain**.

Several quantities are disposable in order to describe damping: one is the time constant of the damping (or **time constant of the envelope**) ϑ of the individual partials. During the length of a time constant, the level of the respective partial drops by 8,686 dB. A vibration with a level dropping off by 60 db within 10 s has a time constant of 1,45 s. The duration of time that it takes a level to drop by 60 dB is – in room acoustics – also called the **reverberation time T_N**. The latter is suitable to describe a damping, as well: the formula $T_N = 6{,}91 \cdot \vartheta$ holds. **Fig. 1.47** shows the course of the levels of the fundamentals (*G#*) measured via the piezo pickup. During the initial second, the time constants differ by a factor of 18.

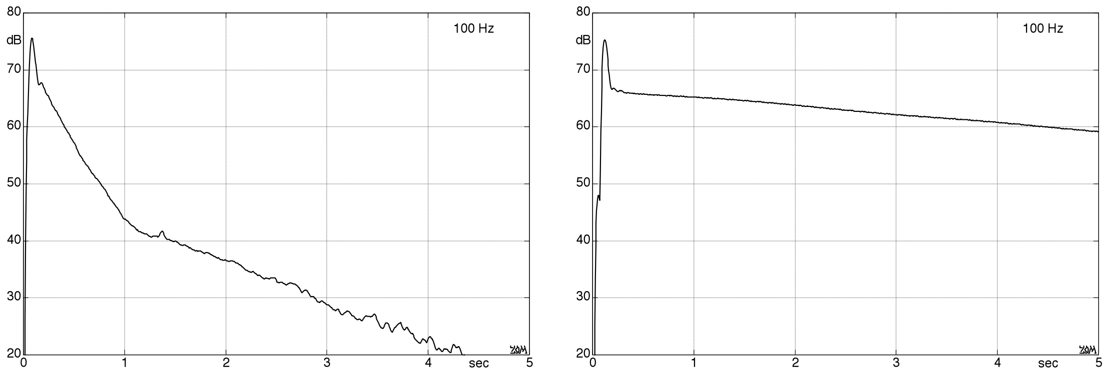

Fig. 1.47: Measurements with Ovation guitars: SMT (acoustic guitar, left); Viper (electric guitar, right).

The following considerations are based on the law of conservation of energy. In the plucking process, the string is given a certain potential energy that is in part dissipated and in part radiated. As an EXAMPLE, a string is to be plucked with 5 mWs; it then decays in different ways. Which sound pressure level is generated at a distance of 1 m if we assume – to begin with – that 100% of the vibration energy is radiated as **sound wave**?

For any exact calculation we would have to know about the beaming – as a simplification let us assume an omni-directional characteristic here. In fact, this assumption is a good approximation for the (quite level-strong) 2nd partial of the E-string [1]. The energy E of the spherical wave [3] is calculated as:

$$E = \frac{4\pi R^2}{Z_0} \int_0^{\infty} p^2(t)\,dt = \frac{4\pi R^2}{Z_0} \cdot \frac{\hat{p}^2}{4} \cdot \vartheta \qquad \text{with } Z_0 = 414 \text{ Ns/m}^3$$

Herein, $p(t)$ is the sound pressure at the distance R = 1m; the integral over the damped vibration was already calculated at the end of Chapter 1.6.1. The equation can be solved for the sound pressure amplitude:

$$\hat{p} = \sqrt{\frac{Z_0}{\pi R^2} \cdot \frac{E}{\vartheta}} \quad \text{in the example} \quad \hat{p} = 0{,}57\,\text{Pa} \quad \text{for } \eta = 100\% \text{ und } \vartheta = 2\,\text{s}.$$

From the (now known) sound pressure, the level can be calculated e.g. for exponential FAST-averaging (**Fig. 1.48, left section**, different ϑ). ◊

$$L(t) = 10\lg\left(\frac{Z_0 E / p_0}{2\pi R^2 (\vartheta - 2\tau)} \cdot \left(e^{-2t/\vartheta} - e^{-t/\tau}\right)\right) \qquad \vartheta \neq 2\tau$$

The time constant ϑ of the damping influences both the maximum value and the speed of decay. The luthier can increase the peak sound pressure level via high mechano-acoustical coupling – the loudness will then decrease more quickly, though. Lower coupling will enable him (or her) to achieve longer sustain, but then the guitars is not as loud. The plucking energy is present only once, after all. Now, if we allow the string to vibrate in two planes, the seemingly impossible is in reach: a loud guitar with long sustain. The top-normal vibration generates a loud attack. The quick decay of this loud attack is “drowned out” after a short time by the more slowly decaying top-parallel vibration.

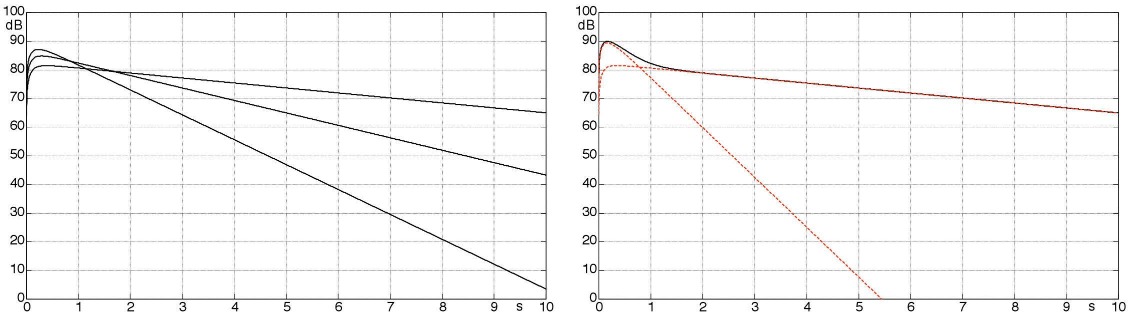

Fig. 1.48: Left: FAST SPL for different degrees of coupling between string and sound field (η = 100%). Right: FAST-SPL for two superimposed orthogonal vibrations (η = 100%). Equal energy.

Fig. 1.48 (right hand section) shows an example with both vibrations being excited with 5 mWs. The quicker decay happens at a time constant of the damping of 0,5 s, the longer decay has a time constant of 5 s. The dashed lines indicated the levels of the individual vibrations. An efficiency of 100% is assumed again for both vibrations.

Of course, in practice an **efficiency** of 100% is not achievable; part of the vibration is converted into caloric energy already within the string, and in the guitar body, as well. Reducing the efficiency to 50% will also reduce the time constant of the decay by half (this may be deduced via the transmission-line equation). The course of the level will then be determined by two parameters: the mechano-acoustical matching, and the dissipation in the guitar (**Abb. 1.49**).

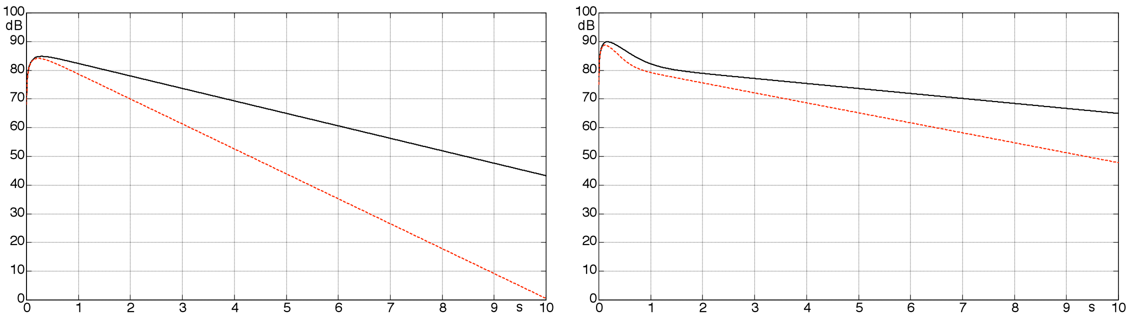

Fig. 1.49: Calculated SPL for an excitation energy of 5 mWs (left) and 2.5 mWs (right). The solid line indicates an efficiency of 100 %, the dashed one an efficiency of 50%.

