2.3 Standing waves

The waves considered so far all featured a direction of propagation: either towards the right i.e. in the direction of increasing z-coordinate (positive propagation speed c), or towards the left (negative c). Such waves are called *propagating waves* or *travelling waves*. They transport **active (wattful) energy**: $E = P \cdot t = F \cdot v \cdot t$. Two superimposed, equal-energy waves running towards each other yield zero energy flux, though. There is reactive energy in the potential spring-energy or in the kinetic mass-energy; however, the mean power value across full periods of the vibration is still zero.

For a transmission line terminated at its end with an infinite bearing impedance Z, it is not possible to feed any energy to the bearing. This is because the velocity at the bearing is always zero: $v = F / Z = F / \infty$. Therefore all of the wave energy arriving at the bearing is reflected – with the amplitudes of the waves running to and from being necessarily equal. The superposition resulting from this is designated **standing wave**. This term holds for every waveshape but is particularly descriptive for sinusoidal waves (**Fig. 2.10**). In the propagating wave, the amplitude is constant and the phase changes as a function of time, while in the standing wave, the phase (as a function of place) remains constant but the amplitude changes over time.

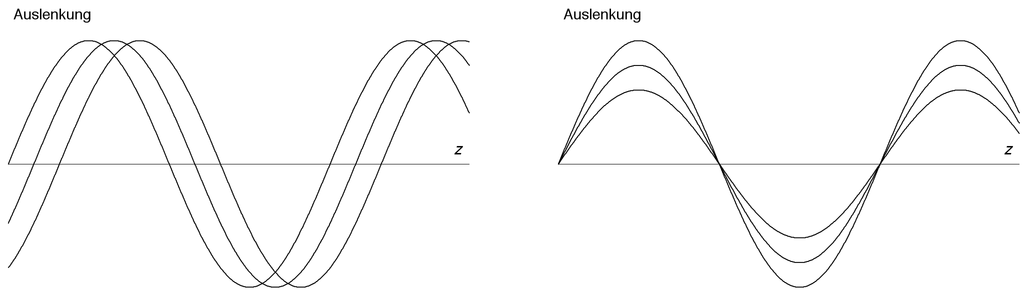

Fig. 2.10: Propagating sinusoidal wave (left); standing wave (right). Along the place-coordinate (z) , the displacement is shown at three consecutive points in time. “Auslenkung” = displacement.

Literature often describes waves running on transmission lines as sinusoidal. For guitar strings, however, we find (at least during the plucking process) a **triangular shape**. At the plucking point, the string is deflected by a transverse force, and for a moment there is (approximately) a triangular string deflection. As soon as the contact between pick (or finger) and string breaks off, two triangular waves run from each other in opposite directions. They are reflected at the string bearings and form – as a sum of all reflections – a standing wave. Instead of reflections we could also define **mirror waves** (see the previous chapter) that run in an unimpeded manner across the bearing points (without reflection). In that model, the **boundary conditions** of the triangular excitation shape, and of the idealized bearing condition of $\xi \equiv 0$ need to be respected. Given the simplifying assumption of lossless propagation and reflection, every wave is reflected an infinite number of times. Therefore an infinite number of mirror waves is required that all run along the string with equal magnitude of the propagation speed. All waves running with a positive c can be combined (superimposed) into *one* summation wave running to the right; the same way all waves running to the left can be combined. The standing wave thus may be described by two summation waves running in different directions.

Since in the present model we assume dispersion-free propagation, the propagation speed *c* is not dependent on frequency. The distance in time between two reflections (of the same event) occurring in the same direction (!) therefore is $T = 2l/c$ for all spectral components. Herein *l* corresponds to the length of the string – it needs to be run through twice until the subsequent reflection occurs e.g. at the right-hand bearing. Given knowledge of the place-function of the excitation, the sum of the waves is easily described with this: its place-periodicity is double the length of the string, and the displacement- and velocity-place-functions are point-symmetric relative to the bearing. At the point in time $t = 0$ both summation waves are identical but run away from each other in opposite directions for $t > 0$. The term **summation wave** indicates the sum of all waves travelling *in the same direction*. The summation wave running towards the left needs to be added to the summation wave running towards the right in order to obtain the actual wave on the string. [Animations can be found at: https:://www.gitec-forum-eng.de/knowledge-base-2/collection-of-animations/].

Fig. 2.11 shows a string deflected in triangular fashion between its bearing points. The top row starts on the left with the initial state. To the right, the two summation waves are depicted – the displacement may be thought as both being combined. At the point in time $t = 0$ the two summation waves are identical, and therefore only *one single* curve can be seen. In the right-hand section of the figure we see a later point in time, with the summation waves having already diverged a bit. The superposition of the two summation waves (second row in the figure) gives the actual course of the displacement – which at the bearing points needs to be zero always (unyielding bearing). In the right-hand graph of the lower row of the figure, several subsequent points in time of the wave propagation are depicted.

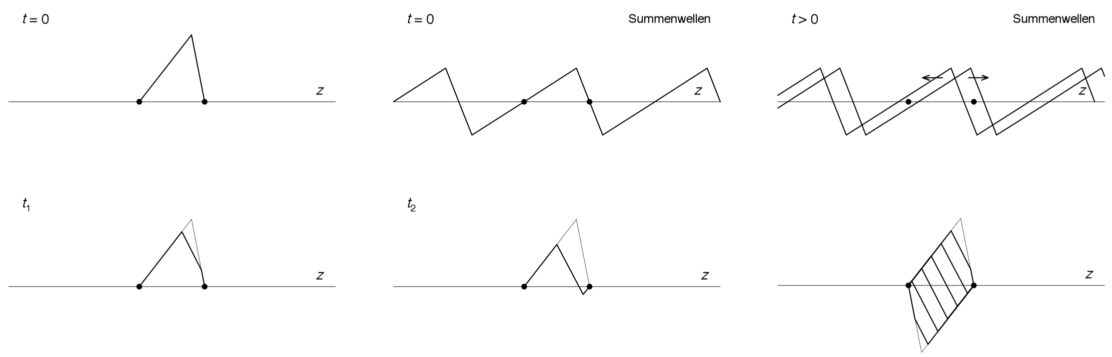

Fig. 2.11: Propagation of a triangular displacement wave. The unyielding string bearings are marked as dots. The wave runs back and forth in a zigzag shape between the dashed end positions.
"Summenwellen" = summation waves.

Books frequently depict string vibrations in a sine-shape – similar to the graphs in Fig. 2.10. These are, however, mono-frequent special cases. The shape of the displacement is – at the moment of excitation – triangular, as shown in Fig. 2.11. The string oscillates back and forth in a zigzag shape; the wave-shape changes over time, though. The damping increases with frequency and blunts the shape, and in addition dispersion occurs (the high frequencies run with a higher propagation speed). These changes in shape are not embraced here; Fig. 2.11 shows a simplification of the basic behavior. The plane of vibration is not considered, either: the vibration of the string is a movement in space, with rotation of the plane of polarization occurring at the bearings. Even with the string plucked e.g. precisely perpendicular to the fretboard, a fretboard-parallel component will emerge over time.

From the place-function of the displacement shown in Fig. 2.11, the place-functions of the velocity and the force may be deduced – and their time-functions, as well. The velocity is the excitation quantity for the magnetic pickup, and the force is the quantity affecting the bearing (as it is processed e.g. by piezoelectric pickups). It was already shown that the velocity results from the *place*-derivative of the displacement – however this holds only for propagating waves and not for standing waves. In the following equation, the propagation speed c needs to be inserted including its sign: for waves running towards the right this is by definition negative, and thus $-c$ becomes positive.

$$v(z)\Big|_{t=t_0} = -c \cdot \frac{d\xi(z)}{dz}\Big|_{t=t_0}$$ Place-function: displacement → velocity

The standing wave therefore is to be dissected – as shown in Fig. 2.12 – into two summation waves. The place-derivative of each of these waves is then multiplied by $-c$. For triangular excitation, the result is depicted in Fig. 2.12: the triangular **displacement** does not oscillate up and down – rather, a zigzag-wave runs back and forth between the triangular border-positions. The **velocity** has the shape of a rectangular impulse that is reflected in opposite phase at the bearings. The **force**-wave has a rectangular shape, as well, but the reflection happens with the same phase here. All three place-functions are standing waves aggregated from two summation waves each. Between the summation waves, a simple conversion ($\xi \Leftrightarrow v \Leftrightarrow F$) is possible, while for the actual aggregated functions (standing waves) a simple correspondence can only by found between the displacement and the force: $F = -\Psi \cdot \partial\xi / \partial z$.

In order to be able to attribute the place-function of the force unequivocally to the displacement, Fig. 2.13 again shows the spring-mass-model. For two conditions, it very nicely demonstrates the triangular displacement, and the rectangular distribution of the (spring-) force.

With the place-functions known, we can now determine the time-functions. Again this holds: the summation-place-functions can be converted into the summation-time-functions with little effort, while for standing waves this is not directly possible. First we will look at the bearing force that results from two summation waves running towards each other. The starting condition ($v \equiv 0$) forces both summation waves $F(z)$ to have an identical shape at the starting point in time ($t = 0$); the bearing condition ($\xi = 0$) forces an odd (point-symmetric) course of the displacement $\xi(z)$ relative to the bearing, and an even (axisymmetric) course of the force $F(z)$, due to the differentiation. Because both axisymmetric summation waves run towards the bearing with equal-amount propagation speed c, the bearing force amounts – at any given point in time – to twice the force acting on the bearing due to a single summation wave. Therefore, the time-function of the bearing force can be determined from the place-function of the force-summation-wave via a simple argument-transformation ($z = ct$), see Fig. 2.14.

The periodic time-function of the bearing force is linked to a spectrum of discrete lines with the fundamental frequency of $f_0 = 1/T$; $T = 2l/c$ is the time-periodicity here. The spectral envelope is an **si-function** [$\mathrm{si}(x) = \sin(x)/x$]; its zeroes result from the partitioning of the place-related displacement: a string partitioning of 4:1 cancels the 5$^{\text{th}}$ harmonic.

Animations at: https:/www.gitec-forum-eng.de/knowledge-base-2/collection-of-animations/

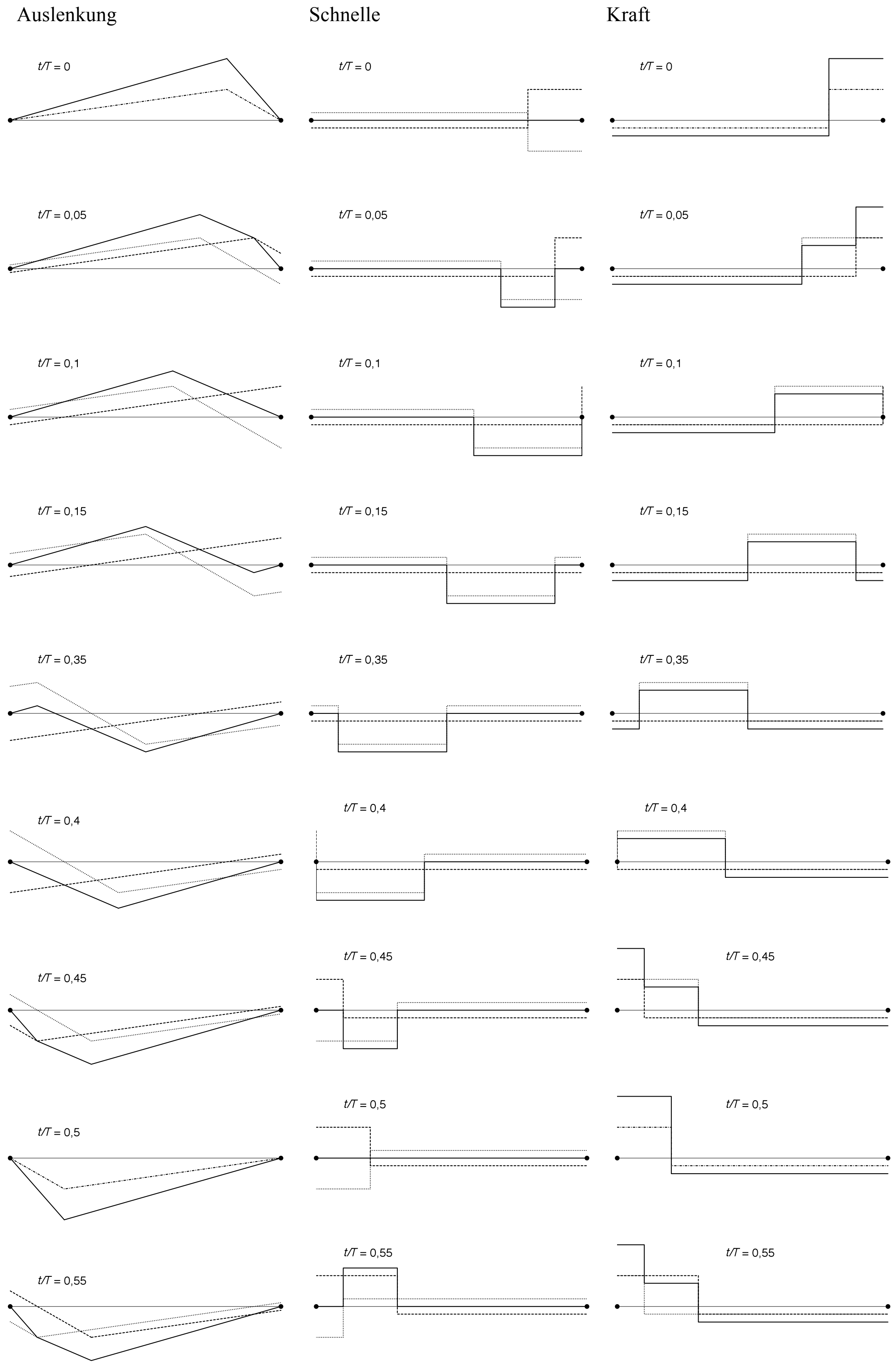

Fig. 2.12: Triangle wave: place-function for 9 different points in time (T = periodicity). The velocity triangle is reflected with opposite phase, the force triangle with the same phase. Direction of propagation: ----→, ←······. "Auslenkung" = displacement, "Schnelle" = (particle) velocity, "Kraft" = force.

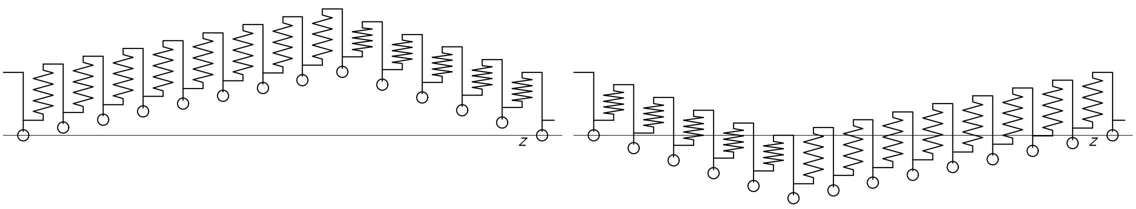

Fig. 2.13: Triangle wave: spring-mass-model (compare to Fig. 2.6). The place-function of the force may be deduced from the deformation of the springs. In the graph on the left, the force left of the salient point is negative, the force to the right is positive (sign convention: compression stress = positive sign). In the right hand graph, the force left of the salient point is positive; to the right it is negative.

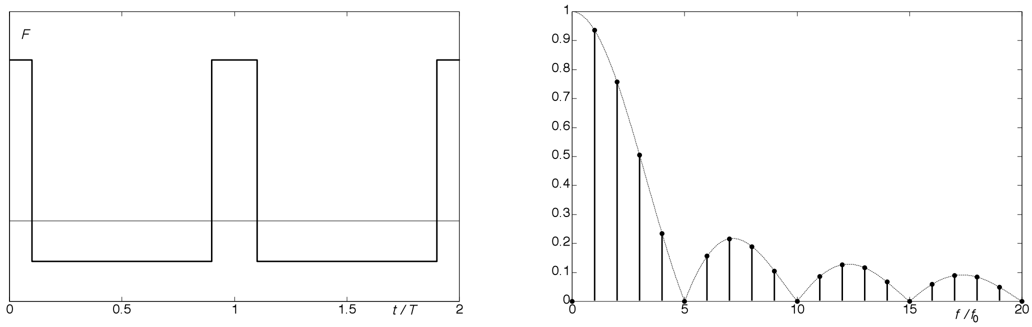

Fig. 2.14: Time-function and magnitude spectrum of the bearing force; triangular displacement similar to Fig. 2.12. The average of the progression of the force over time is zero, and the DC-component in the spectrum thus is zero, as well. Integer multiples of the quintuple of the fundamental frequency are cancelled if the distance between bridge and plucking point is 1/5th of the length of the string (graph on the left).

Fig. 2.15 presents the result of a voltage measurement. The E_4-string of an Ovation (EA-86) was plucked using a plectrum, with the built-in piezo pickup serving as sensor. The shape of the voltage is basically rectangular (i.e. a pulse) – the superimposed vibrations are effects of the dispersive wave propagation. We can interpret the piezo pickup in a simplified fashion as a force-voltage converter transforming the wave forces acting in the bridge into a correspondingly proportional electrical voltage. The duty factor of the pulse corresponds to the division-ratio of the plucking point on the string (32:32, 51:13).

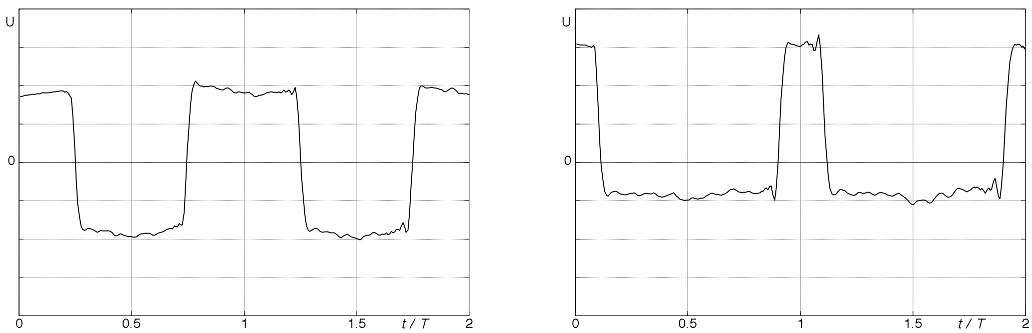

Fig. 2.15: Electrical voltage measured in the piezo pickup built into the bridge. Basically, the shape is rectangular – cause of the vibrations are resonances and dispersive wave propagation. Distance from plucking point to the bridge: 32cm (left), 13cm (right); String length = 64 cm.

At the point in time of the plucking action (assumed to be at t = 0), the actual **velocity of the string** is zero for all points on the string – the string starts off displaced but at rest. For the two summation waves, opposite signs follow: $v_R(z, t=0) = -v_L(z, t=0)$. Here, the index marks the direction of the propagation: R = running right, L = running left. Moreover, the actual velocity is always zero at the bearings (assumed to be immobile), and therefore the summation waves need to be point-symmetric relative to each other for t = 0. Consequently, $v_R(z, t) = -v_L(-z, t)$ is valid for all points in time. From these two conditions it follows that both summation waves are even functions for t = 0: $v_R(z, t=0) = v_R(-z, t=0)$, $v_L(z, t=0) = v_L(-z, t=0)$.

For the electric guitar, the string velocity is the input quantity for the **magnetic pickup**. Determining the spectrum is more complicated than for the piezo pickup because velocity sensors cannot be operated at the bridge. Typically, the magnetic pickup is located below the string at 3 – 15 cm away from the bridge, this distance being designated z_{TS} while the corresponding delay time is termed τ_{TS}. To determine the string velocity above the pickup, we start from the triangular string displacement, and require several transformations. The actual displacement is dissected into two summation waves, and the local derivative yields the place-function of the velocity. Then, an argument-transformation ($z = z_0 - ct$) yields (from the place-function) the time-function, with the time-delay τ_{TS} corresponding to a phase-shift in the frequency domain. At the bridge, the actual velocity is the result of two components that always sum up to zero due to the above mentioned symmetries: $v_\Sigma(t) = v(t) - v(t)$. At the position of the pickup, the delay time needs to be considered with different signs: $v_\Sigma(t) = v(t + \tau_{TS}) - v(t - \tau_{TS})$. With the displacement law of the Fourier-transform, this results in:

$$\underline{V}_\Sigma(j\omega) = \underline{V}(j\omega)\cdot e^{j\omega\tau} - \underline{V}(j\omega)\cdot e^{-j\omega\tau} = \underline{V}(j\omega)\cdot 2j\cdot\sin(\omega\tau) \qquad \text{mit } \tau = \tau_{TS} = z_{TS}/c$$

Herein, $\underline{V}_\Sigma(j\omega)$ is the velocity spectrum of the string at the location of the pickup; this spectrum results from the velocity spectrum $\underline{V}(j\omega)$ of a summation wave via multiplication with a sine-function. The summation wave of the velocity features a harmonic spectrum of discrete lines with the zeroes in the si-shaped spectral envelope determined by the plucking location – as it was for the bearing force. This spectrum is to be multiplied with the above-mentioned sine-function, the zeroes of which are determined by the position of the pickup. **Fig. 2.17** shows the velocity spectra for an E_2-string. Depending on the pickup placement and the place of plucking, a characteristic, sound-determining envelope results (shown in red in the figure).

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Fig. 2.17: Level-spectrum of the actual string velocity at the place of the pickup. The place of plucking is 11 cm from the bridge; the pickup is located at a distance of 15 cm (left) and 5cm (right) from the bridge. Scale length (length of the string) = 66 cm. f_0 = 82,4 Hz.

 Translation by Tilmann Zwicker