2.4 Transitory processes

Systems theory describes linear, time-invariant (LTI-) systems via their **impulse response**. In fact, the impulse response $h(t)$ is a *system*-quantity; it may however also be seen as a *signal*-quantity found at the output of the system that in turn is excited at its input with a (Dirac-) impulse $\delta(t)$. Using, instead of the Dirac-impulse, its particulate integral over time, the output of the system yields the particulate integral of the impulse response: this is the **step response**. Dirac-impulse and step are idealizations that, in reality, occur merely approximately. As a pre-consideration, let us excite the string with a **force step**: a transverse force acting externally on the string changes its value from 0 to F at the point in time of $t = 0$, with the string at rest (not deflected) for any time t < 0. It is unimportant for the model consideration how such a force-step can be realized, but it is important that F remains constant – and in particular that it does not depend on the displacement. The string bearing is immobile at *one* position ($z = 0$), and the other (right-hand) bearing is very far away. At the distance d from the bearing at $z = 0$, the **external force** F acts on the string (**Fig. 2.18**).

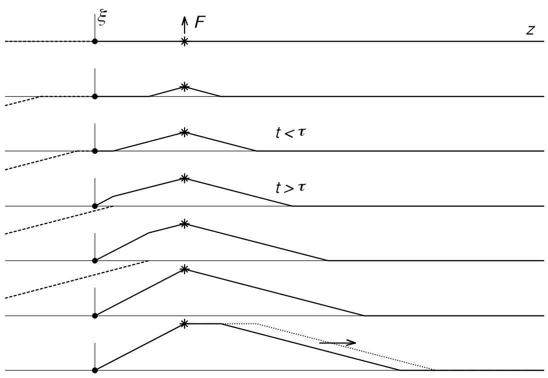

Fig. 2.18: Place-function of the displacement. Shown from top to bottom are 7 subsequent states. The immobile bearing is given by a dot; a constant external force acts at the place marked by a star. For the first 5 graphs, the mirror wave arriving from the left is indicated by the dashed line; for the last two graphs it is not shown. The further course of the wave is represented as a dotted line in the bottom graph.

The wave impedance Z_W is defined via the (mechanical) string data. As long as no reflection has arrived at the excitation point (star), Z_W describes the quotient between force F and velocity v. Since the excitation point is, however, loaded by *two* transmission lines (to the left and to the right), the input impedance is doubled i.e. it is $2Z_W$ (seen from an external point of view). In considerations of analogy with an electrical line, we need take into account that the *F-I*-analogy results in reciprocal impedances: impedance ↔ admittance. Imprinting a constant force at the location of the star will generate a transverse movement with the constant velocity: $v = F/(2Z_W)$. The **reflection** is considered via a mirror wave arriving from the left; it reaches the location of the star after the time $\tau = 2d/c$ (c = propagation speed). For $t > \tau$, the quotient between F and v is not determined by Z_W anymore, because there are now two waves superimposed at the location of the star. The two counteracting velocity-waves interact such that the point of the string marked by a star changes its velocity from v to zero at $t = \tau$. This point remains at a fixed displacement for $t > \tau$. The displacement $\hat{\xi}$ at this location may be calculated:

$$\hat{\xi} = \tau \cdot v = \frac{2d \cdot F}{c \cdot 2Z_W} = \frac{d \cdot F}{\Psi}$$

Maximum displacement at the location of the star, $t \geq \tau$

The parallelogram of forces yields the same value if the tension force of the string Ψ, and the transverse force are F, are formulated orthogonally: $\hat{\xi}/d = F/\Psi$.

The point in time of $t = \tau$ separates two different processes: during $t < \tau$ the transient process (an aperiodic movement) takes place. For $t > \tau$, the stationary final state between the bearing (symbolized by the dot) and the point where the force is applied (symbolized by the star) is reached.

In the case that (as is shown in Fig. 2.18) the right hand bearing is very far away, a slope (indicated as dotted line in the figure) runs to the right without perturbation. The section of the string between the (right-hand) bearing and the position given by the star stops moving at the point in time of τ; it then remains at rest. However, if reflections can happen on the right-hand side, as well, a continuously vibrating **standing wave** results. Still, this model does not simulate the plucking process because in the latter the force does not jump from 0 to F but from F to 0. Given LTI-conditions, though, an $F \rightarrow 0$ jump may be seen as the sum of a negative force-step and a force constant at all times:

$$F = F_1 + F_2; \qquad F_1 = \begin{cases} 0 & t < 0 \\ -\hat{F} & t > 0 \end{cases}; \qquad F_2 = \hat{F}\ .$$

The boundary conditions now are: for negative time a constant force $\hat{F}$ acts on a point of the string – the string is displaced but at rest. At the point in time $t = 0$, the force jumps from $\hat{F}$ to 0 with an oscillation starting that is superimposed onto the triangular displacement. The initial situation ($t < 0$) is shown in **Fig. 2.19**. The external force $\hat{F}$ (constant over time) finds its counter-forces in the bearing forces F_L and F_R. While the signs of the string-internal forces and the external forces require some getting-used-to, they are consistent. For positive time $t >$ 0, the external force $\hat{F}$ vanishes – from this point in time the two bearing forces thus need to be void of any mean value (**Fig. 2.20**).

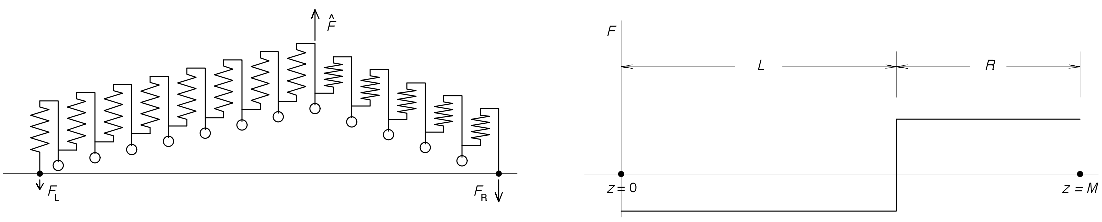

Fig. 2.19: Spring-mass-model for $t < 0$ (left), and corresponding string-internal force-place-function.

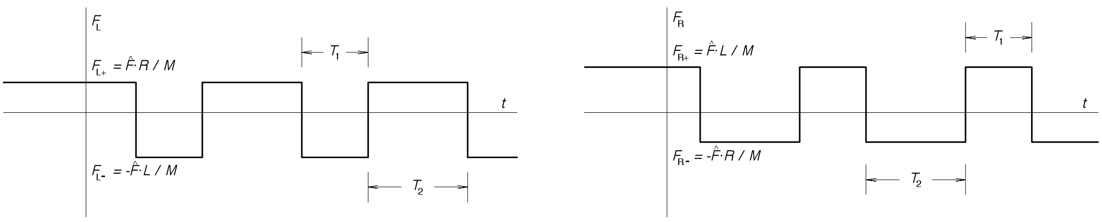

Fig. 2.20: Time-function for the bearing forces; at $t = 0$, the excitation $\hat{F}$ jumps to zero. The signs of the bearing forces are defined comparably among each other: if the string is displaced in the ξ-direction, a counter-force needs to act at both bearings; their direction is indicated with an arrow. The string-length is M – it is divided into a left-hand (L) and a right-hand (R) section. $T_1 = TR/M$, $T_2 = TL/M$, $T = 1/f_G$.

