2.5 Calculation of reflections

In Chapter 2.2, we had introduced a model of mirror-waves in order to describe reflections of waves. In it, the wave under consideration runs across the bearing (and disappears), and at the same time a mirror wave running in the opposite direction emerges from the bearing. As an alternative to these two waves propagating in an unperturbed fashion, it may be expedient to look at only *one single* wave that is reflected at the bearing according to certain criteria. This reflection model gives advantages in particular for the type of modeling of the string that uses delay units.

2.5.1 The reflection factor

Every propagating wave (travelling wave) transports energy: in the electrical transmission line this is the energy of the magnetic and electric field, while in the mechanical line it is kinetic and potential energy. The mean values of the mechanical energy calculate as:

$$E_k = dm \cdot v^2/2\,; \quad E_p = dn \cdot F^2/2 \qquad \xrightarrow{F = v \cdot Z_W} \qquad E_k = E_p$$

The two transported (mean) energies are equal at each place of the transmission line. As the wave arrives at the end of the string, this energy cannot disappear into nothingness; it is either coupled into the bearing (and transported further there, or dissipated) or it is (fully or partially) reflected.

All bearings show complicated **bearing impedances**. The bearing impedance is anisotropic i.e. depends on the plane of vibration, and it is dependent on frequency. The compliance is the inverse of the complex bearing impedance and is defined as a complex **admittance**:

$$\underline{Y} = \underline{v}/\underline{F} = G + jB \quad \text{Admittance} = \text{conductance} + j \cdot \text{susceptance}$$

$$\underline{Z} = \underline{F}/\underline{v} = R + jX \quad \text{Impedance} = \text{resistance} + j \cdot \text{reactance} = 1/\text{admittance}$$

An unyielding, rigid bearing (small admittance, high impedance) can absorb forces but does not allow for movement; the compliant bearing behaves conversely. Strings are anchored in relatively unyielding bearings. For the electric guitar only, the bearings may be totally rigid – in the real world, such an ideal is of course not possible. If the bearings on an acoustic guitar were fully unyielding, they could (due to $v = 0$) not receive any energy from the string, and could not transmit it further to eventually radiate sound.

The bearing impedance (or admittance) connects the two field quantities of force and (particle) velocity; their product is the power P. The requirement for continuity demands $F_{\text{string}} = F_{\text{bearing}}$ and $v_{\text{string}} = v_{\text{bearing}}$. On the string, the quotient F/v is equal to Z_W for the propagating wave, but at the bearing, this quotient may take on any value. At first, this appears to be a contradiction. If a 2-N-force-wave runs through a transmission line of a wave-impedance of 1 Ns/m, the velocity is 2 m/s. As this wave now encounters a bearing of a bearing-impedance of 10 Ns/m, the bearing cannot fully absorb the wave energy. The bearing “extracts” from the arriving wave that part of the energy that matches the bearing impedance in terms of the F- and v-components. The remainder of the energy is “sent back”.

Therefore, *two* waves (incoming and reflected) running in opposite directions are superimposed at the bearing and at every point of the string. Force and velocity thus result from the sum of two values. The wave running in the opposite direction at the plucking location has to be considered including its reflection – and the subsequently generated reflections, as well. All waves are reflected after having run the length of the string, i.e. more and more waves superimpose. The sum of all superimposed waves results in the **steady-state condition** that may be calculated via the tools offered by network analysis. Calculating the line impedance $Z(z)$ for this steady-state condition at any arbitrary point z will not yield the wave impedance (at least not for the general case).

At this point, the wave "sent back" in the above example is unknown. In order to calculate it, we formulate the force $F(z)$ and the velocity $v(z)$ acting at each point of the line as a sum of two waves♣. The waves $F_h(z)$ and $v_h(z)$ running towards the bearing are given; knowing one of the two is sufficient; the other can be calculated from it. The reflected waves $F_r(z)$ and $v_r(z)$, while also linked via Z_W, are yet unknown. The bearing impedance delivers the missing condition, because at the bearing point (e.g. at $z = 0$) the quotient of $F(z = 0)$ and $v(z = 0)$ is equal to the **bearing impedance** Z_L. As has already been the case a number of times, the **sign** springs a surprise: at the right-hand bearing, $Z_L = F/v$ holds, and at the left-hand bearing, we have $Z_L = -F/v$. This reversal of the sign is easiest seen in Fig. 2.5: a left-hand bearing can be generated by making $F_1 = 0$; the left-hand mass is now removed, and the formula indicated as "law of inertia" carries a minus-sign. Similarly, $F_2 = 0$ yields a plus-sign. The wave impedance includes its peculiarity in terms of the sign, too: for waves running towards the left, Z_W is negative, for those running towards the right, it is positive (Chapter 2.1). Superimposing the waves running back and forth we would have to do the math with two different wave impedances. However, for the following calculations Z_W is **strictly positive** – for the waves running to the left we insert a minus-sign. In the below calculation we consider a wave running ("hither") towards the left onto the left-hand bearing ($z = 0$), and a wave reflected towards the right:

$$F_h = -v_h \cdot Z_W; \qquad F_r = v_r \cdot Z_W; \qquad Z_W = \sqrt{\Psi \cdot m'} \quad > 0$$

$$F_L = F_h + F_r; \qquad v_L = v_h + v_r; \qquad Z_L = -F_L / v_L$$

$$(v_h + v_r) \cdot Z_L = -(F_h + F_r) = -(-v_h + v_r) \cdot Z_W \qquad \Rightarrow \qquad \frac{v_r}{v_h} = \frac{Z_W - Z_L}{Z_W + Z_L}$$

The ratio of the complex amplitudes within the back-and-forth-running wave is the complex **reflection coefficient** r. It is dependent on the wave impedance Z_W and on the bearing impedance Z_L:

$$r_v = \frac{v_r}{v_h} = \frac{Z_W - Z_L}{Z_W + Z_L} \qquad r_F = \frac{F_r}{F_h} = \frac{v_r \cdot Z_W}{-v_h \cdot Z_W} = -r_v \qquad \text{Reflection coefficient}$$

There are three interesting special cases: for $Z_L = Z_W$ (**matching condition**), the reflection coefficient becomes zero: the wave is not reflected and the bearing absorbs the whole of the wave energy without reflection. For $Z_L \to 0$, the reflection coefficient of the velocity becomes +1: the velocity wave is completely reflected with the same phase, and the force wave is completely reflected with opposite phase.

♣ F, v, and Z are complex; we make do without the underscoring here.

For $Z_L \to \infty$, the reflection coefficient of the velocity becomes −1: the velocity wave is completely reflected with opposite phase, and the force wave is completely reflected with the same phase. This is the case of the unyielding bearing where the velocity of the string is always zero. Of course, a guitar string must not be operated with $r = 0$ – otherwise a "periodic" vibration would never come into being. With $r = \pm 1$, the vibration would never decay – at least within the idealizations underlying here.

In Chapter 1.6, investigations regarding the decay process of the string vibration were introduced. It the vibration of an E_2-string decreases (strictly exponentially) e.g. by 60 dB within 12 s, it decays by 0,06 dB per 12 ms (1 period), corresponding to 0,7%. The reflection coefficient therefore is 0,993 per period. Since the wave on the string is reflected twice per period, this absorption of 0,7% needs to be divided up between bridge and nut (or fret), e.g. 0,3% at the nut/fret and 0,4% at the bridge. Typically, a refection coefficient of close to 1 is found.

Given strictly **real bearing impedance**, the reflection coefficient is real because Z_W is real, as well. For a real r, the phase shift between the original and the reflected wave is either 0° or 180°. In contrast to the reflection at an imaginary bearing impedance, the amplitude of the reflected wave is now smaller than that of the original wave. For a guitar string, the bearing impedance Z_L is large compared to Z_W, yielding the following as an approximation:

$$r_v = \frac{Z_W - Z_L}{Z_W + Z_L} = -\frac{1 - Z_W / Z_L}{1 + Z_W / Z_L} \approx -(1 - Z_W / Z_L) \cdot (1 - Z_W / Z_L) \approx -(1 - 2 Z_W / Z_L)$$

A negative-real reflection coefficient indicates that the velocity-reflection happens with the opposite phase. If the real part of the reflection coefficient is not zero, active energy flows into the bearing points (**dissipation**, string damping). It makes no difference for the string whether this energy is radiated from the guitar body, or is converted directly into heat within the bearing – the drained energy is not available anymore as vibration energy.

The other extreme would be a purely imaginary bearing impedance as it is formed by a mass or a spring. Even if the bearing is composed of several masses and springs, *at any one single frequency* there will be either one inert or one stiff bearing impedance. For a **purely imaginary bearing impedance**, numerator and denominator of the reflection coefficient are complex conjugate; the absolute value of r therefore is 1. That is <u>exactly</u> 1! The waves running back and forth are phase-shifted relative to each other, but the absolute value is conserved: the vibration energy does not decrease. However, since the phase of propagating waves changes as a function of the place (wave equation), a phase-shifted reflection may be seen as non-phase-shifted reflection from another place. We can imagine that the wave is reflected without phase shift but at a small distance behind the bridge, with the phase shift resulting from this detour corresponding to the actual reflection. Depending on the sign it may be necessary to shift this imagined reflection place ahead of the bridge. The same holds for the nut (or fret). The effective string length may therefore differ from the geometric one: depending on the bearing impedance, and on the frequency, the length may be longer or shorter. This influences the frequency of the partials:

> A springy bearing extends the effective length of the string, and it decreases the vibration frequency; the softer the spring, the lower f is. A mass-loaded bearing shortens the string and decreases the frequency; the lighter the mass the higher f is.

The reflection coefficient of the real string has both a real and an imaginary component, with both depending on the frequency. The real part causes the damping of the string, while the imaginary part has a detuning effect. In addition, string-internal mechanisms need to be considered – the present chapter is dedicated to the loss-free transmission line.

EXAMPLE: a tensioned string (L = 64 cm, $\rho = 8{\cdot}10^3$ kg/m^3, S = 0,5 mm^2, Ψ = 100 N) is suspended immovably on one side and springily on the other with s = 10.000 N/m, $s \neq s(f)$.

From this follows: Z_W = 0,633 Ns/m, c = 158,1 m/s, f_G = 123,5 Hz (without influence of the spring). Considering the elastic edge suspension, the fundamental frequency f_G decreases:

$$Z_L = s/j\omega = -j\cdot 12{,}89\,\text{Ns/m} \qquad r = \frac{Z_W - Z_L}{Z_W + Z_L} = -0{,}9952 + 0{,}0979\,j = e^{j\cdot 174{,}4^\circ}$$

The absolute value of the reflection coefficient is 1, the angle is smaller than 180° by 5,6°. Running through a full length of the string, the phase of the wave is changed by 180°; a phase delay of 5,6° corresponds to a path-length of 2 cm. The one-sided elastic suspension effectively lengthens the string by 2 cm, decreasing the fundamental frequency to 119,8 Hz♣. The relative detuning is identical for all harmonics (disregarding the dispersion).◊

2.5.2 A resonator serving as bearing for the string

Any real bearing of a string needs to feature not only components behaving like springs, but also masses – and that makes bearing resonances unavoidable. At the resonance frequencies, the reactances (or conductances) compensate each other. Impedance and admittance are exclusively real. At all other frequencies, impedance and admittance remain complex [3].

As an example, a loss-free spring/mass-system will be investigated in the following. The impedance of its bearing computes to:

$$Z_L = j\omega m + s/j\omega \qquad \omega_r = \sqrt{s/m} \qquad f_r = \omega_r/2\pi$$

For $\omega = \omega_r$ the impedance of the bearing becomes zero (no force despite movement), while for $\omega < \omega_r$ the bearing acts like a spring (spring-controlled). For $\omega > \omega_r$ it acts inert (mass-controlled). Below resonance, a string coupled to the bearing is in effect elongated. Above resonance, it will in effect be shortened. Even assuming the string to be dispersion-free, the frequencies of the partials are not laid out harmonically anymore: below the resonance frequency of the bearing, the frequency of the partials decreases, and above the resonance frequency of the bearing, it increases. The reflection coefficient for the velocity is:

$$r_v = \frac{Z_W - Z_L}{Z_W + Z_L} = \frac{Z_W - j(\omega m - s/\omega)}{Z_W + j(\omega m - s/\omega)} = -\frac{p^2 m - pZ_W + s}{p^2 m + pZ_W + s} \qquad p = j\omega$$

The frequency dependence of the reflection coefficient $r_v(j\omega)$ leads to a 2nd-order rational function. The even numerator- and denominator-potencies are identical, while the odd ones have an inverted sign. Numerator and denominator thus are complex conjugate relative to each other. This kind of frequency dependence is termed **all-pass function.**

♣ Real bearings are much stiffer; with them the detuning is smaller.