The reflection coefficient of the real string has both a real and an imaginary component, with both depending on the frequency. The real part causes the damping of the string, while the imaginary part has a detuning effect. In addition, string-internal mechanisms need to be considered – the present chapter is dedicated to the loss-free transmission line.

EXAMPLE: a tensioned string (L = 64 cm, $\rho = 8\cdot 10^3$ kg/m^3, S = 0,5 mm^2, Ψ = 100 N) is suspended immovably on one side and springily on the other with s = 10.000 N/m, $s \neq s(f)$.

From this follows: Z_W = 0,633 Ns/m, c = 158,1 m/s, f_G = 123,5 Hz (without influence of the spring). Considering the elastic edge suspension, the fundamental frequency f_G decreases:

$$Z_L = s/j\omega = -j\cdot 12{,}89\,\mathrm{Ns/m} \qquad r = \frac{Z_W - Z_L}{Z_W + Z_L} = -0{,}9952 + 0{,}0979j = e^{j\cdot 174{,}4°}$$

The absolute value of the reflection coefficient is 1, the angle is smaller than 180° by 5,6°. Running through a full length of the string, the phase of the wave is changed by 180°; a phase delay of 5,6° corresponds to a path-length of 2 cm. The one-sided elastic suspension effectively lengthens the string by 2 cm, decreasing the fundamental frequency to 119,8 Hz♣. The relative detuning is identical for all harmonics (disregarding the dispersion).◊

2.5.2 A resonator serving as bearing for the string

Any real bearing of a string needs to feature not only components behaving like springs, but also masses – and that makes bearing resonances unavoidable. At the resonance frequencies, the reactances (or conductances) compensate each other. Impedance and admittance are exclusively real. At all other frequencies, impedance and admittance remain complex [3].

As an example, a loss-free spring/mass-system will be investigated in the following. The impedance of its bearing computes to:

$$Z_L = j\omega m + s/j\omega \qquad \omega_r = \sqrt{s/m} \qquad f_r = \omega_r/2\pi$$

For $\omega = \omega_r$ the impedance of the bearing becomes zero (no force despite movement), while for $\omega < \omega_r$ the bearing acts like a spring (spring-controlled). For $\omega > \omega_r$ it acts inert (mass-controlled). Below resonance, a string coupled to the bearing is in effect elongated. Above resonance, it will in effect be shortened. Even assuming the string to be dispersion-free, the frequencies of the partials are not laid out harmonically anymore: below the resonance frequency of the bearing, the frequency of the partials decreases, and above the resonance frequency of the bearing, it increases. The reflection coefficient for the velocity is:

$$r_v = \frac{Z_W - Z_L}{Z_W + Z_L} = \frac{Z_W - j(\omega m - s/\omega)}{Z_W + j(\omega m - s/\omega)} = -\frac{p^2 m - pZ_W + s}{p^2 m + pZ_W + s} \qquad p = j\omega$$

The frequency dependence of the reflection coefficient $r_v(j\omega)$ leads to a 2nd-order rational function. The even numerator- and denominator-potencies are identical, while the odd ones have an inverted sign. Numerator and denominator thus are complex conjugate relative to each other. This kind of frequency dependence is termed **all-pass function.**

♣ Real bearings are much stiffer; with them the detuning is smaller.

The magnitude of an all-pass function is 1, and the phase shifts by $n \cdot \pi$ for $0 : f : \infty$, n being the order of the all-pass function. For $f = 0$, $r_v = -1$ holds: the velocity wave is reflected with opposite sign. For $f = f_r$ we obtain $r_v = +1$; for $f \to \infty$ we again get $r_v = -1$.

Therefore, having a resonator terminating the transmission line has the effect of an additional phase shift. Natural vibrations (partials) occur at those frequencies where the phase shift for a full travel-path on the string (2L; back and forth) is an integer multiple of 2π. Assuming dispersion-free wave propagation on a fully clamped-down string, partials at integer multiples of the fundamental frequency result. However, if a bearing acts as a resonator, an additional phase shift is introduced that generates (in our example) an **additional partial**. For resonators of higher order, several additional partials occur.

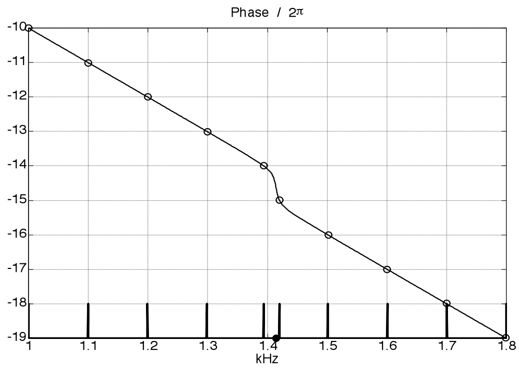

Fig. 2.21: Phase shift along a full travel path along the string. One string bearing is configured as a resonator resonating at 1,415 kHz. An additional natural frequency is the result of the narrow-band additional phase shift.

Fig. 2.21 shows the phase shift occurring for a string vibrating at a fundamental frequency of 100 Hz and a full travel path (double the string length). The phase is negative as it is customary for delays in recent literature. One string bearing is configured as a resonator with a resonance frequency of 1,415 kHz (dot on the abscissa). At the bottom of the graph, the frequencies of the partials are indicated with bars. The partial at 1,4 kHz is substantially detuned downwards by the bearing resonance, and an additional partial is generated at 1,42 kHz. All other de-tunings are too small to be recognizable in the figure.

The spectral derivative $-d\varphi/d\omega$ yields the **group delay** (Chapter 1.3.1). The slope of the phase function is virtually constant with the exception of the range around the bearing resonance. Thus, the group delay is also generally constant – only in the range of the bearing resonance it becomes longer. This leads to a warping in the spectrogram (Fig. 1.8).

2.6 Line losses

Ideal masses and springs store energy but do not dissipate them as heat. These elements are therefore termed “loss-less”. In contrast, any real string also features friction-resistances that irreversibly convert the vibration energy into caloric energy. Line theory considers these energy losses via distributed, differentially small resistances. It is insignificant for the model whether the losses are due to mechanical friction in the string (inner damping), or result from the string *directly* radiating sound energy (i.e. without detour through the guitar body).

