As a **bottom line**, we may state: inner damping and radiation losses may be disregarded as long as merely the wave propagation along short sections of the string is discussed. When analyzing vibrations of longer duration, we find – in electric guitars – damping mechanisms having a greater effect towards the higher frequencies (Chapter 7.7), and additional frequency-selective absorptions (e.g. resonances of the bridge). For acoustic guitars, we need to expect substantial absorptions in the low-frequency range, as well, since a non-negligible share of the vibration energy is fed to the bearings (bridge, frets).

2.7 Dispersive bending waves

The simple transmission line theory assumes place-independent wave impedance and frequency-independent propagation speed. However, the transversal waves of the guitar string propagate in a dispersive fashion, i.e. with frequency-dependent speed. The high frequencies run faster than the low ones (Chapter 1.3.1). The reason is the bending stiffness that increases the transverse stiffness, the latter in turn depending on the tensioning force.

Modeling the string as a dispersive transmission line takes much effort and is not always necessary. In most cases, only two or three points on the string are of interest (nut/fret, bridge, and point of plucking). Possibly, the position of the pickup also needs to be added in. It is easy to model the parts of the line between the discrete points via all-passes (Chapter 2.8). However, if precise description of the reflection conditions is required, we need a more detailed model. The simplest solution is found for steady-state (mono-frequent) partials: propagation speed and wave impedance are only weakly dependent on the frequency. For narrow-band considerations they may in fact be assumed to be constant. Transient processes extend across a frequency *range*, though; in such cases we need to apply frequency-dependent quantities.

We had introduced a simple element for modeling the dispersion-free string in Abb. 2.5. As characterizing quantities, force and velocity were sufficient (both quantities being signal-, place- and time-dependent). However, the rigidity of the real string requires that in addition to the (transverse) **force $\boldsymbol{F}$**, a place- and time-dependent **bending moment $\boldsymbol{M}$** is specified, and also that we introduce an **angular speed $\boldsymbol{w}$**. This gives us a frequency-dependent phase delay (Fig. 1.6). The dispersive line element cannot be described as a quadripole (two-port network); rather, we need to specify a **four-port network** (octapole) [11]. The input quantities of the latter are F_1, M_1, v_1, w_1; its output quantities are F_2, M_2, v_2, w_2. Because the transverse dimensions of the string are small relative to the wavelength, we may disregard shear deformations and rotational inertia moments (Euler-Bernoulli theory for beams). Thus, the length-specific **mass $\boldsymbol{m'}$**, the length-specific **compliance $\boldsymbol{n'}$**, and the **bending stiffness $\boldsymbol{B}$** remain as the system quantities (inside the four-port network).

The rigid string features *two* **wave impedances** $Z_F = F/v$ and $Z_M = M/w$, and *two* wave powers $P_F = Fv$ and $P_M = Mw$. *Two* bearing impedances each are active at both string bearings (nut/fret, bridge), and in addition the four signal quantities may be intercoupled in each bearing. For example, the edge-force may generate an edge-moment, or a displacement will necessarily lead to torsion. Since all these relationships appear depending both on frequency and direction, simplifications and approximations are indispensable.

Waves of lower order (fundamental and low-frequency harmonics) are not influenced much by the rigidity. The effective overall rigidity is practically only determined by the tensioning force Ψ, with the dispersion remaining insignificant (Fig. 1.4). However, for higher-order partials the influence of the rigidity may not be ignored anymore – especially for the low strings. The (overall-) rigidity as it is significant for the higher partials consists of two components: a frequency-*in*dependent portion caused by the tensioning force, and a frequency-dependent portion caused by the rigidity. The differential equation for the bending wave is of 4^{th} order; we therefore require four boundary conditions, and four independent fundamental solutions are possible. As had been the case for the rigidity-free string, a **wave** running forward and a **wave** running backward appear in the longitudinal direction, but in addition, an exponential **fringe field** is superimposed close to the bearings. Fortunately, this fringe field decays already at a short distance, and further away from the bearings we may therefore do the math with only one wave type. Without the fringe field, we obtain a simple coupling between F, v, M, w: knowing one of these four quantities suffices to describe the other three. *One single* wave equation is good enough to describe the string vibration (in one plane); we need a frequency-dependent wave number $k(\omega)$ for it, though.

This simplification is not valid for the description of reflections, though, because the latter indeed occur especially within the fringe zone. In this context, “fringe” refers to the beginning and the end of the string, and not the mantle-surface of the cylinder. Within the fringe zone, we need to formulate – in addition to the wave equation – a fringe field with its own wave number k', designated **fringe-field number**. Although in the fringe field the signal quantities F and v are still linked via Z_F (as M and w are linked via Z_M), F may take on any value independently of M (and the other way round) due to the fringe field. While in the dispersion-free string the reflection coefficient depends only on the ratio of wave impedance / bearing impedance, two wave impedances and two bearing impedances (per bearing each!) define the **reflection coefficients** in the stiff string. Thus, it is (at least theoretically) possible to reflect the Fv-wave entirely at the bearing, and to entirely absorb the Mw-wave. This does, however, not mean that there is no Mw-wave running in the reverse direction: the fringe field will take care of the existence of an Mw-wave already at a short distance – the energy necessary for this is “withdrawn” from the Fv-wave.

Within the abundance of all the reflection conditions possible in every vibration plane, there are some special cases that may be easily analyzed:

- Open end of the string: the string ‘dangles in the air’; its end cannot absorb any transverse force F, nor any moment M. While this seems rather lacking in practical relevance, it may appear at resonance.
- Clamped string: transverse velocity v and angular speed are zero.
- Guided end: angular speed w and transverse force F are zero.
- Supported string: transverse velocity v and moment M are zero.

The real string bearing is not represented in any of the above special cases. This is because the string does normally not end at the bearing but is guided across it. Often the string rests in a small notch that permits for line-shaped contact only. This inhibits any transverse movement but allows for forces, angular movements and moments. If we interpret this bearing as a large blocking-mass, it will reflect Fv-waves but not Mw-waves! For the extreme case of a string featuring a stiffness that is only determined by the bending stiffness (beam), a barrier-mass reflects 50% of the incident wave energy – the other 50% are coupled as a bending wave into the section of the string beyond the bearing. In the other extreme case ($B = 0$), though, 100% of the energy is reflected.

In order to assess the significance of the bending stiffness, let's look at the following **model case**: the string is supported by a knife-edge bearing not allowing for any lateral movement. The string also continues indefinitely beyond this first bearing, the other bearing has ideal reflecting characteristics. The percentile energy-portion transmitted beyond the bearing is shown for an A_2-string in **Fig. 2.22**. At low frequencies, the bending stiffness is negligible; the energy is almost completely reflected. However, already from middle frequencies a significant percentile is coupled across the (immobile!) bearing. On the other side of the bearing, we do not see a pure *Mw*-wave; rather, the fringe field again takes care of generating a combination of *Fv*- and *Mw*-waves.

Of course, a real string cannot extend indefinitely; it ends after a few centimeters at the tuner ("machine head"), in the string retainer, in the body, or wherever else there is space to attach it. Fig. 2.22 clearly indicates that it does make a difference where and how the string is fastened, though. The string-part beyond the bearing may indeed tap considerable vibration energy if it has corresponding length, forming a coupled resonator. Still, the power-percentile shown in Fig. 2.22 is not necessarily lost at each and every reflection. The share of energy coupled across the bearing may itself be reflected e.g. at the tailpiece, run back to the bearing, and then is once more coupled across the bearing into the main part of the string. Also, the real string does not have a line-shaped contact to the bearing: via a contact area (groove), not just a pure transverse force may be received but a moment as well. Some bridge/nut-combinations are deliberately (?) designed with larger contact surface, or directly as clamping-devices. For the latter, a further model-case will be discussed at the end of the chapter.

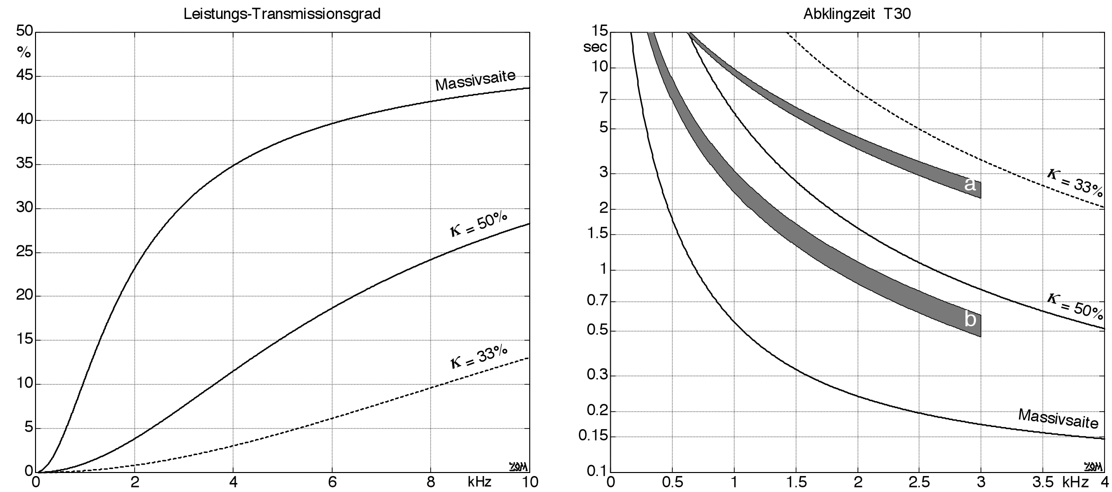

Fig. 2.22: Degree of power transmission ("Leistungs-Transmissionsgrad") for an A_2-string (40 mil); one end is clamped, the other borne on a knife-edge. On the right, the decay time ("Abklingzeit") of the partials purely due to the transmissions is given for a level-decrease of 30 dB via with three calculated lines (decay time T30, Chapter 7.6.3). The ratio of core diameter to outer diameter is κ = 50% (——), or 33% (-----), or 100% for the solid string ("Massivsaite").

The grey areas show results of measurements (A_2, 40 mil, κ = 50%) taken on the stone table. For **a**, the string was clamped at both ends, for **b** one end was clamped and the other supported: remaining string length is 30 m, weakly damped.

Case a fits well to the "orientation line" presented in Chapter 7 (Fig. 7.66); in addition to the bearing, string-internal damping mechanisms are at work, as well.

Case b should be compared to the 50%-line above it. This (calculated) line considers only the absorption occurring at the support-type bearing. In contrast, the measurements (grey area) also include string-internal damping mechanisms, and the absorption at the other bearing (clamp).

In an E_2-string, the losses due to the transmission are even larger.

To calculate the conditions for vibration and reflection, the string is divided into small **cylindric sections** of the length *dz*. At rest, the circular separation planes (cross-sections) are perpendicular to the *z*-axis. As the string is excited, the cross-sectional surfaces remain flat but are not in parallel anymore due to the bending moments: they form an angle of curvature. The laws of motion, inertia, and strength result in a partial differential equation for the rigid string (for detail see the supplement):

$$\Psi \cdot \frac{\partial^2 \xi}{\partial z^2} - B \cdot \frac{\partial^4 \xi}{\partial z^4} = m' \cdot \frac{\partial^2 \xi}{\partial t^2}$$ Differential equation for the string

The differential equation (DEQ) is a *partial* one because it includes the derivatives for both place *z* and time *t*; it is *linear* because the variables of transverse displacement ξ, place *z*, and time *t* are present in the first power only; it includes *constant coefficients* because the system quantities of tension force Ψ, bending stiffness *B*, and length-specific mass *m'* are not dependent on *z* and *t* (idealized); and it is homogenous because it does not comprise an external excitation.

B and *m'* are determined from the material data and the geometry of the string; the tension force Ψ results from the required fundamental frequency f_G. Any function $\xi(z,t)$ that will satisfy the DEQ is a **solution** for it. According to DANIEL BERNOULLI, the solution for sinusoidal movement is formulated as a product including a purely time-dependent and a purely place-dependent factor:

$$\xi(z,t) = \hat{\xi} \cdot e^{j(\omega t + \varphi)} \cdot e^{-jkz} = \underline{\xi} \cdot e^{-jkz}$$ Solution approach

The first factor $\underline{\xi}$ includes the angular frequency ω and the initial phase φ; a partial differentiation regarding the time *t* becomes a multiplication with $j\omega$. The second factor holds the wave number *k*; a partial differentiation regarding the place *z* becomes a multiplication with $-jk$. Introducing the corresponding derivatives into the DEQ yields:

$$-k^2 \Psi \underline{\xi} - k^4 B \underline{\xi} = -\omega^2 m' \underline{\xi}$$ Characteristic equation

The characteristic equation may be cancelled by $\underline{\xi}$ (the case $\xi \equiv 0$ being trivial). This yields a conditional equation for *k* that includes only a dependency on the system quantities. Because this equation is of 4th order, there are four independent solutions for which four independent boundary conditions need to be specified. In terms of the solution approach, two *k*-values are real, and the exponent therefore is imaginary ($-jkz$). This describes a **sinusoidal wave** running to the left or to the right, respectively. The other two values for *k* are imaginary, and the exponent thus is real – describing an exponentially increasing/decaying **fringe field** originating from the string bearing. Only the decaying fringe field is of practical importance. The general equation of motion is a superposition of the two wave equations and the equation of the decaying fringe field:

$$\xi(z,t) = \underline{\xi}_1 \cdot e^{+jkz} + \underline{\xi}_2 \cdot e^{-jkz} + \underline{\xi}_3 \cdot e^{-k'z}$$ general solution

The time-dependency is found in the three independent complex amplitudes $\underline{\xi}_i$, the frequency is identical for all three components.

In the following, we will consider a string ($z \geq 0$); the (left-hand) bearing is located at $z = 0$. The expressions

$$\xi_A(z,t) = \underline{\xi} \cdot e^{jkz}, \qquad \underline{\xi} = \hat{\xi}_A \cdot e^{j(\omega t + \varphi)}; \qquad \text{Excitation}$$

describe a sinusoidal wave running left towards the bearing (ξ = displacement). A part of its energy is reflected at $z = 0$, the remainder is transmitted:

$$\xi_R(z,t) = \zeta \cdot \underline{\xi} \cdot e^{-jkz}, \qquad \xi_T(z,t) = \psi \cdot \underline{\xi} \cdot e^{jkz}; \qquad \text{Reflection, transmission}$$

In the general case, reflection coefficient ζ and transmission coefficients ψ are complex. On the considered section of the string ($z \geq 0$), three displacements are superimposed:

$$\xi(z,t) = \underline{\xi} \cdot \left(e^{jkz} + \zeta \cdot e^{-jkz} + \gamma \cdot e^{-k'z} \right) \qquad z \geq 0$$

γ represents the complex fringe-field coefficient. Beyond the bearing (i.e. in the range of the transmission) two displacements are superimposed:

$$\xi(z,t) = \underline{\xi} \cdot \left(\psi \cdot e^{jkz} + \delta \cdot e^{k'z} \right) \qquad z \leq 0$$

Here, too, a fringe field with a different wave number k' and fringe-field coefficient δ is generated (in addition to the transmitted share). Fringe fields and waves are functions of the place (z) and the time (t). The dependency on time is described by $\underline{\xi}$ with ω as circular frequency; the place-dependency is described via the **fringe-field numbers** k and k'. For the propagating waves, $k = 2\pi / \lambda$ is reciprocal to the wavelength λ. The fundamental frequency f_G is the lowest eigenfrequency (natural frequency) of a string; λ_G corresponds to double the length of the string – in the E_2-string this is about 1,3 m for 82,4 Hz. Partials in the range around 10 kHz therefore have a wavelength around 1 cm ($\lambda_n = \lambda_G / n$). This still much exceeds the string diameter – we thus may do the math using approximations. For the high strings, these conditions are met to an even higher degree.

From the fringe-field number k', a limit distance $z_g = 1/k'$ may be estimated; it indicates at which distance the fringe field has decayed to 1/e. Since the characteristics are those of a flexural wave, the calculations require somewhat more effort (in particular for the wound strings). **Fig. 2.23** shows typical values of z_g.

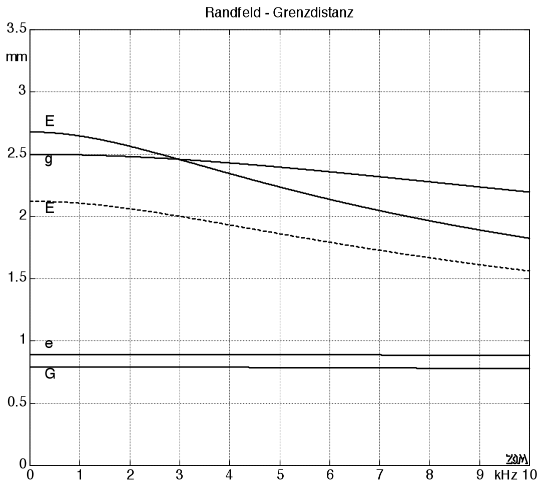

Fig. 2.23: Limit value z_g of the fringe field (in mm)
E_2-string: 53 mil, κ = 0.4
E_2-string: 42 mil, κ = 0.4 ----
G_3-string: 24 mil, κ = 0.5
g_3-string 20 mil, plain
e_4-string: 12 mil, plain

"Randfeld – Grenzdistanz" = limit value z_g of the fringe field

Running towards the bearing, the excitation wave is specified by its amplitude $\hat{\xi}_A$ and its frequency *ω*. For *this same* wave, *F*, *M* and *w* are defined via the system quantities *B*, *m'* and *Ψ*, and so are the wave impedances Z_F and Z_M, as well as the velocity *v* with $v = \partial\xi/\partial t$. The bearing (at $z = 0$) is – to begin with – defined by its two **bearing impedances** $Z_{FL} = F(0)/v(0)$ and $Z_{ML} = M(0)/w(0)$. Considering the string to be a linear system, there is the superposition of three oscillations in the range of $z \geq 0$: the given excitation wave ($\hat{\xi}_A$), the reflected wave (*ζ*), and the fringe field (*γ*). At first, *ζ* and *γ* are two unknown quantities; however, they may be calculated via the two bearing impedances.

The system quantities of the string are tension force *Ψ*, length-specific mass $m' = \rho S$, and bending stiffness $B = ES^2/4\pi$. Herein defined are ρ = density, S = cross-sectional surface, and E = Young's modulus. For wound strings, it is predominantly the core that defines the bending stiffness; the densities of core and winding may differ [appendix]. From these quantities, the w**ave number *k*** and the fringe **field number *k'*** may be calculated:

$$k = \sqrt{\frac{\Psi}{2B}\left(\sqrt{1+4B\omega^2 m'/\Psi^2} - 1\right)} \qquad k' = \sqrt{\frac{\Psi}{2B}\left(\sqrt{1+4B\omega^2 m'/\Psi^2} + 1\right)}$$

Both *k* and *k'* are system quantities, as well, i.e. they are signal independent. The rigid, tensioned string can be transformed into two borderline cases by varying *B* and *Ψ*: for $B = 0$ we obtain the dispersion-free string (fully flexible), and for e $\Psi = 0$ we get the cantilever (without any tensioning force). The wave numbers are calculated as:

$$k \to \omega\sqrt{\frac{m'}{\Psi}} = \frac{\omega}{c} \qquad k' \to \infty \qquad B \to 0 \qquad \text{String without bending stiffness}$$

$$k \to \sqrt{\frac{\omega}{B}\sqrt{Bm'}} = \frac{\omega}{c(\omega)} \qquad k' \to k \qquad \Psi \to 0 \qquad \text{Beam without tensioning force}$$

The phase velocity *c* is frequency-dependent for $B \neq 0$, and for $B \to 0$ it is constant. The wave-reflection coefficient *ζ* is calculated as:

$$\zeta = \frac{\left(Bk^2 + \Psi + \omega Z_{FL}/k\right)\left(\omega Z_{ML}/Bk + jk'/k\right) + \left(Bk'^2 - \Psi + j\omega Z_{FL}/k'\right)\left(1 + \omega Z_{ML}/Bk\right)}{\left(Bk^2 + \Psi - \omega Z_{FL}/k\right)\left(\omega Z_{ML}/Bk + jk'/k\right) - \left(Bk'^2 - \Psi + j\omega Z_{FL}/k'\right)\left(1 - \omega Z_{ML}/Bk\right)}$$

The formulas now do start be become rather lengthy – but they still do not fully describe the bearing. In fact, the simplification based on two bearing impedances Z_{FL} and Z_{ML} (as it is sometimes found in literature) is not always sufficient. In the general case, a coupling between the transversal quantities *F* or *v*, and the bending quantities *M* or *w*, respectively, may occur; the bearing impedance in that case receives the form of a matrix, and moreover an additional coupling term. Using this, a formal description is still explicitly possible, but the practical use of the formulas is increasingly limited because the individual bearing quantities cannot be measured with sufficient accuracy anymore. The vast diversity of bearing parameters forces to simplify – and it calls for the question how well these simplifications fit in the individual case.

The string vibration may be approximated in different ways:

a) The simplest approximation describes the string without its bending stiffness. The partials are positioned harmonically, and the propagation velocity is frequency-independent. To describe the bearing and the reflection, a single bearing impedance Z_{FL} is sufficient – it may be determined e.g. with an impedance head. For the fundamental f_G and the lowest partials, this approximation is adequate in many cases, but already in the middle frequency range we recognize clear deviations between calculation and measurement (Figs. 1.5, 1.7).

b) The calculation of the partials with consideration of the bending stiffness represents an easily obtainable improvement. On average, the actual spreading of the partials is quite well met. Considering moreover also the dilatational waves (Fig. 1.17) yields a useful approximation for the level spectrum.

c) In order to calculate the decay processes, the bearing impedances need to be known. For very light strings, we may disregard the bending stiffness, but for heavier strings knowledge of the bearing impedance Z_{FL} is required besides knowledge of the bearing impedance Z_{ML}.

d) The supposed "fully comprehensive" description of the bearing quickly degenerates into a confusingly extended system of equations: in two orthogonal vibration planes, we need to define three bearing impedances each – not to forget additional coupling impedances between the two planes. In addition, the impedance of the longitudinal wave should be borne in mind, again including mode coupling to the two orthogonal transversal waves. Presumably, a torsion wave on the string may be ignored – but this assumption is still under scrutiny: for the bowed string, the torsion wave is significant. Since all bearing- and coupling impedances depend (in some cases strongly) on the frequency, a confusing multitude of parameters results.

The next **example** shows that the bending stiffness of the string can make for problems even at low frequencies although the tensioning stiffness should in fact be predominant in this frequency range. For the calculation, we assume an idealized support bearing that is immobile in the transverse direction. The transversal velocity therefore is zero at this bearing. However, for bending processes that are coupled to the angular speed, this bearing is supposed to feature a moment of inertia (**blocking mass**). Due to the (material- and geometry-dependent) bending stiffness of the string, and due to the inertia of the bearing, a resonance may arise that (depending on the circumstances) may absorb a significant part of the vibration energy, or may couple this energy into the section of the string beyond the bearing (**total transition**). For a very small or a very large blocking mass, the resonance frequency will appear at very high or very low frequencies – it will then not cause any disturbance. However, given a corresponding dimensioning, resonances can appear in the middle frequency range, as well. Such resonances are not generally undesired – possibly, the luthier seeks to obtain a somewhat stronger absorption exactly in that frequency range. However, to use this in a targeted manner, the (frequency-dependent) moment of inertia at the bearing would have to be known – this poses problems for the instrumentation.

The following calculation circumvents the instrumentation issue and defines idealized bearing parameters; the approach does not orient itself on a special realization. In the discussion of the cone-parameters following later, we will again look into this subject matter, and we shall dive some more into the details.

As EXAMPLE, we look at a wound E-string of a diameter of 46 mil. It rests on a bearing such that its lateral movement is zero; Z_{FL} thus becomes infinite. The reflection coefficient:

$$\zeta = \frac{\left(Bk^2 + \Psi + \omega Z_{FL}/k\right)\left(\omega Z_{ML}/Bk + j\,k'/k\right) + \left(Bk'^2 - \Psi + j\omega Z_{FL}/k'\right)\left(1 + \omega Z_{ML}/Bk\right)}{\left(Bk^2 + \Psi - \omega Z_{FL}/k\right)\left(\omega Z_{ML}/Bk + j\,k'/k\right) - \left(Bk'^2 - \Psi + j\omega Z_{FL}/k'\right)\left(1 - \omega Z_{ML}/Bk\right)}$$

therefore is simplified. It reads:

$$\zeta = \frac{\left(1 - j\,k'/k\right)\omega Z_{ML}/Bk \; + \; 1 + (k'/k)^2}{\left(1 + j\,k'/k\right)\omega Z_{ML}/Bk \; - \; 1 - (k'/k)^2}$$ Reflection coefficient

The bending impedance Z_{ML} of the bearing is negative because the excitation wave runs towards the left:

$$Z_{ML} = -(j\omega\Theta + W)$$ Θ = moment of inertia of the bearing, W = bearing resistance

Using this, the complex reflection coefficient can be calculated:

$$\zeta = -\frac{Bk - \omega(j\omega\Theta + W)/(1 + j\,k'/k)}{Bk + \omega(j\omega\Theta + W)/(1 - j\,k'/k)}$$

Together, the bending stiffness B and the moment of inertia Θ form a resonance that can be located e.g. in the range of the middle frequencies (**Fig. 2.24**). With a suitable choice of W, total absorption is possible within a narrow frequency range. Such extreme cases may not be expected in typical string bearings, but it is still clearly evident that the bending stiffness can have effects at middle and low frequencies, too. ◊

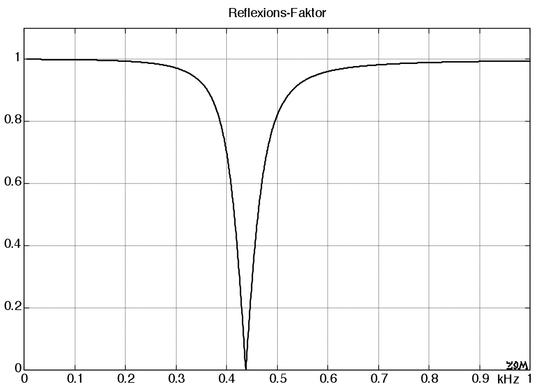

Fig. 2.24: Magnitude of the reflection coefficient of an E_2-Saite, 46 mil, core/outer diameter 50% ($\kappa = 0.5$). The bearing is unyielding in the transversal direction, but has a moment of inertia Θ towards bending stress. $\Theta = 4{,}2{*}10^{-8}$ kgm^2 corresponds to the rotation moment of inertia of a steel ball of a diameter of 10 mm. $W = 1{,}07{*}10^{-5}$ Nsm. "Reflektions-Faktor" = reflection coefficient

Finally, let us look again at the calculation of the transmission coefficient ψ. Given known excitation and known bearing impedance, ψ can be calculated; conversely, an unknown bearing impedance (preferable Z_{ML}) may be calculated if ψ is known. For the bearing that is immobile in the transverse direction, and given a fully flexible string, the transmission is zero – the vibration energy is entirely reflected. Strings that are not ideally flexible can, however, transmit part of their vibration energy across such a bearing (Fig 2.22).

The calculation of the **transmitted part** assumes that a flexural wave with the transversal velocity $\underline{v}$ propagates in the main section of the string. The transverse velocity $v(z = 0)$ at the bearing is supposed to be zero by definition (ideal knife-edge bearing). We see the angular speed $w(z = 0)$ as the coupling quantity; it is identical on both sides of the bearing. The following equation describes v as it occurs in the main section of the string:

$$v(z,t) = \underline{v} \cdot \left(e^{jkz} + \zeta \cdot e^{-jkz} + \gamma \cdot e^{-k'z} \right) \qquad z \geq 0$$

From this, the place-derivative ($w = \partial v / \partial z$) yields the angular speed:

$$w(0) = jk\underline{v} \cdot \left(1 - \zeta + j\gamma\, k'/k\right) \qquad \text{Angular speed at the bearing}$$

For the knife-edge bearing ($Z_{FL} = 0$), the reflection coefficient ζ contained herein results from:

$$\zeta = \frac{\left(1 - j\,k'/k\right)\omega Z_{ML}/Bk \;+\; 1 + (k'/k)^2}{\left(1 + j\,k'/k\right)\omega Z_{ML}/Bk \;-\; 1 - (k'/k)^2} \qquad \text{Reflection coefficient}$$

The ideal knife-edge bearing does not have any bending impedance Z_{ML}. However, the flexural wave arriving at the bearing still does meet a bending impedance: the one of the string extending beyond the bearing ($z < 0$). This impedance is:

$$Z_{ML} = \frac{-Bk}{\omega} \cdot \frac{1 + (k'/k)^2}{1 + j\,k'/k} \qquad \text{Input impedance of the remaining section of the string}$$

The bearing impedance Z_{ML} is negative, because the excitation wave runs *towards the left* to the bearing ($z > 0$). Using Z_{ML}, the (complex) reflection coefficient is simplified:

$$\zeta = \frac{-1}{1 - j\,k'/k} \qquad \text{Reflection coefficient of the knife-edge bearing } (z>0)$$

With this result, the fringe-field coefficient γ is also defined for the knife-edge bearing:

$$\gamma = \frac{-1}{1 + j\,k/k'} \qquad \text{Fringe-field coefficient of the knife-edge bearing } (z>0)$$

Using the above, we can now calculate the angular speed present at the bearing: $w(0) = jk\underline{v}$. However, the progressive wave does not simply travel across the bearing being unimpressed: directly at the far side of the bearing we have $w(-0)$ consisting of the ψ-part of the progressive wave, and the δ-part of the fringe field. The fringe field has decayed at a small distance ($z < 0$), though, and only the ψ-part of the (transmitted) wave running away from the bearing remains.

Given $w(0)$, the remaining section of the string is now excited in the range $z < 0$ (transmitted part); the wave running away and the fringe field superimpose here:

$$v(z,t) = \underline{v} \cdot \left(\psi \cdot e^{jkz} + \delta \cdot e^{k'z} \right) \qquad z \leq 0$$

At $z = -0$, the transversal velocity needs to be zero, too (knife-edge bearing), thus $\delta = -\psi$ holds.

The absolute scaling is calculated using the angular speed $w(0)$ of the bearing:

$$w(0) = j\,k\underline{v} = j\,k\underline{v}\left(\psi - j\delta\, k'/k\right) \qquad \Big\} \qquad \psi = \frac{1}{1 + j\,k'/k}$$

ψ represents the complex **transmission coefficient** – it states which part of the excitation wave runs across the bearing. Given a cantilever without any tensile strain ($\Psi = 0$), $k'/k = 1$ holds. The transmitted amplitude portion amounts to 70%, and the transmitted energy portion is 50%. The other 50% of the energy are reflected. In a guitar string, the tension force Ψ dominates, and thus the reflected portion is larger (Fig. 2.22).

We may not entirely ignore the **coupling across the bridge**, though, as shown by the following experiment. For a semi-solid guitar (Gibson ES-335 TD with trapeze tailpiece), the strings of which continue for 10 cm beyond the bridge to the tailpiece, the E_2-string was set in motion by tapping it between the bridge and the tailpiece, and then immediately damped again. By this, the section of the string between bridge and nut was set in motion, as well, and sustained audibly. However, tapping the string directly above the bridge there is practically no excitation – the transverse impedance of the bridge is indeed very high.

The coupling across the bridge is also pointed to by this experiment: the decay time (sustain) of the ES-335 TD was established for the E_2-string, using third-octave bandwidth. Subsequently, the palm of the hand was placed on the section of the string between bridge and tailpiece, damping it. Again, sustain was measured (for the section of the string between bridge and nut), and it indeed was shorter across the whole frequency range.

Neither experiment provides absolute proof: the sting is supported at the nut, as well, and excitation or damping could have been present here, too. Therefore we carried out a supplementary experiment on the **vibration test rig**: a solid steel wire of 0.7 mm diameter and 13,3 m length was stretched between two bearings each with the shape of mono-pitched roof. A **laser beam** samples the transversal velocity of this "string" at 4 mm ahead of one of the bearings. Beyond the bearing the string continues for 65 mm to where it is fastened (i.e. this is the remaining section of the string). Between the bearing and the measuring point of the laser, the string is hit with a small drop hammer providing an impulse-shaped excitation. The transverse displacement over time is shown in **Fig. 2.25**: once with un-damped remaining section of the string, and next to that with damped remaining section of the string. The bending-coupling is not pronounced very much but it is still clearly visible.

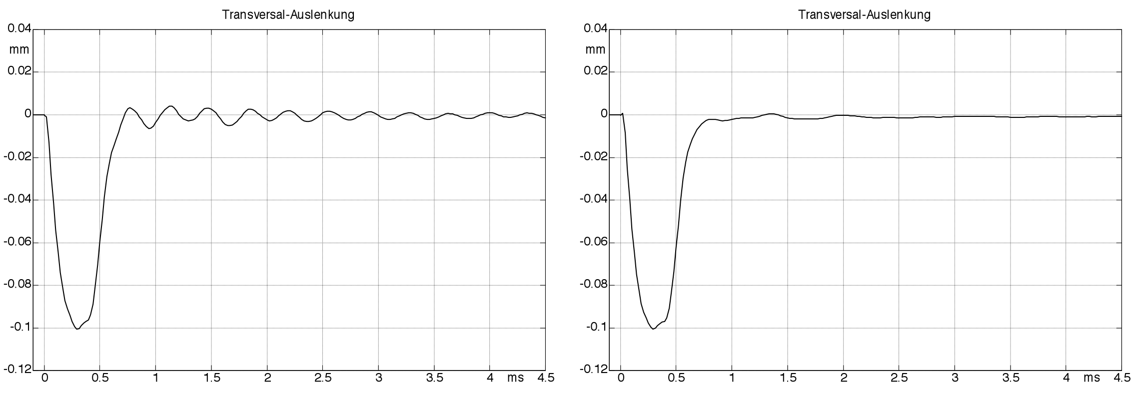

Fig. 2.25: Transverse displacement ("Transversal-Auslenkung") without (left) and with (right) damping of the remaining section of the string (Chapter 1.4).