2.8 The generalized transmission-line model

The guitar is part of a signal-processing system generating sound from the movement of a plectrum (pick). With the input quantities of plectrum-force and plectrum-velocity, and the output quantities of bearing force and bearing velocity (in an acoustic guitar), or pickup voltage and pickup current (in the electric guitar), respectively, the string is a subsystem of the guitar. In Chapter 1.5 we had defined the plucking process as imprinting a force step with the effect that a special square wave runs back and forth on the string. This (more of less) periodic repetition of the excitation signal may be very nicely described with signal-flow diagrams, as they are also used in the context of digital FIR-/IIR-filters. It is not a problem that the signals in digital filters are usually time-discrete and discrete-valued, while the signals on the string are time- and value-continuous. In the simple transmission-line model, only the delay times occurring between the string bearings are emulated via delay lines. Conversely, plucking point and pickup position may be arbitrarily chosen.

2.8.1 Ideal string, bridge pickup

The following signal flow diagrams **SFD** (block diagrams) represent the signal processing via arithmetical operations. The basic operations are delay, summation, subtraction, and multiplication with a constant. The graphs do not give any indication of the source- and load-impedances and must not be confused with a circuit diagram.

A transverse force jumping to zero at the time $t = 0$ is defined as the excitation signal for the string. This force step runs in both directions from the plucking point; its phase velocity is c. The delay time necessary to reach bridge or nut, respectively, depends on c and the distance that needs to be covered. At the end of the string, each force step is reflected – here, we need to distinguish between $r_{bridge} = R$ and $r_{nut} = r$. Thereafter, both force steps circle in a recursive loop with an overall delay time of $T = 2L/c$. **Fig. 2.26** shows the corresponding SFD:

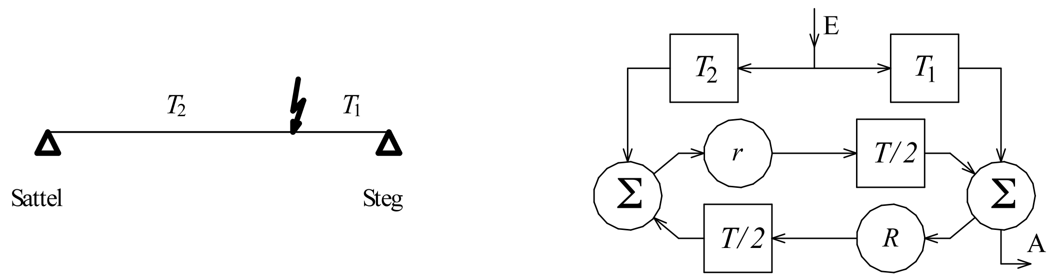

Fig. 2.26: Signal flow diagram (SFD) for non-dispersive string vibration. T_1 and T_2 are delay times from the plucking point to the bridge ("Steg") and the nut ("Sattel"), respectively; R is the reflection coefficient at the bridge, r is the reflection coefficient at the nut, $T/2$ is the delay time between bridge and nut, or nut and bridge, respectively. E = input, A = output (bridge).

The SFD shown in Fig. 2.26 differs from the ideal string in one significant aspect: the impulse created by the plucking runs back and forth on one and the same string, while in the SFD, the paths in the two directions manifest themselves in two separate, serially connected signal branches. Still, the signal processing is identical, and in both cases one cycle includes *two* reflections.

By repositioning of single delays, the SFD can be reshaped to result in a ladder network of three systems (**Fig. 2.27**):

- A basic delay T_1, modeling the delay time from point of plucking to the bridge.
- A recursive system with the delay time T, modeling the string vibration maintained via the reflections (IIR- and AR-filter respectively)
- Am interference filter with a delay difference of $2T_2$, modeling the shaping of the sound color via the point of plucking (FIR- and MA-Filter, respectively). For any one reflection at the nut /(or fret), $r \approx -1$ holds.

This representation has the main advantage that the "plucking"-filter (FIR-filter) and the section of the generator (IIR-filter) are considered independently from each other in separate stages. Assuming un-damped, loss-free vibrations ($R \cdot r = 1$), the IIR-filter (operating just shy of self-oscillation) generates – after impulse excitation – a periodic signal. Obligatorily, there is a matching harmonic line spectrum with the frequency distance of the lines equal to the fundamental frequency of the string.

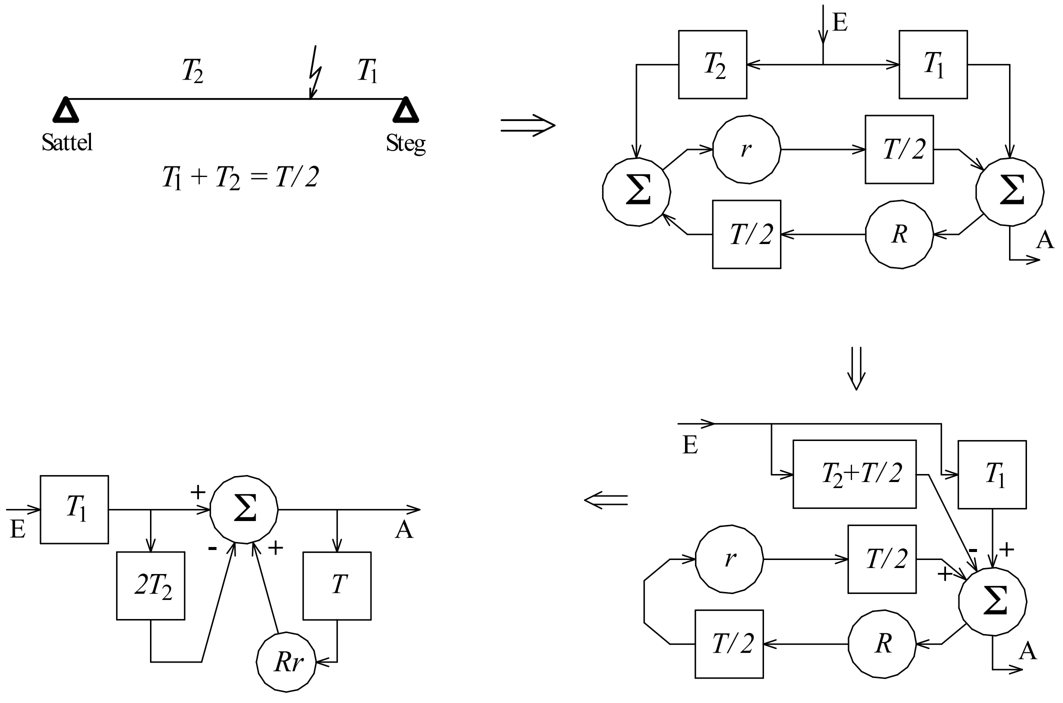

Fig. 2.27: Rearranged signal flow diagram (only a single signal path string → bridge). The sequence of the FIR-filter ($2T_2$) and the IIR-filter (T) is permutable (commutative mapping in the linear system). "Sattel" = nut, "Steg" = bridge.

Rearranging the FIR-delay time is done with $T_1 + T_2 = T/2$, resulting in:

$$(T_2 + T/2) - T_1 = T_2 + T/2 - (T/2 - T_2) = 2T_2$$

Using simple methods known from signal processing [e.g. 5], we can now derive from the SFD shown in Fig. 2.27 the behavior regarding frequency. If we take, as excitation signal, a short impulse (idealized a Dirac) periodically repeated in the IIR-filter, a spectrum with equidistant lines of constant height results. This spectrum is filtered as it runs through the subsequent systems, i.e. it is modified.

A pure signal delay by a constant delay time (e.g. T_1) only changes the phase spectrum but not the magnitude spectrum. We will ignore this basic delay since it is immaterial for the following considerations whether or not the output signal arrives a few milliseconds later. However, the delay time in the FIR-filter must not be ignored since here two signals are superimposed that are delayed with respect to each other – with the resulting frequency-selective amplifications and cancellations (comb-filter). The sequence of FIR/IIR, or IIR/FIR, respectively, must not be interchanged.

The filter effect of the **comb-filter** is extensively described in literature; we will only cover it in short here. The temporal input signal of a delay line arrives at the output after a delay (generally: T_x), the spectrum of the input signal is to be multiplied with the transfer function to yield the spectrum of the output signal. The transfer function $\underline{H}$ of a (pure) delay line with the delay T_x is:

$$\underline{H}(j\omega) = e^{-j\omega T_x}; \qquad \omega = 2\pi f$$ Transfer function of a delay line

In a comb-filter, delayed signal and un-delayed signal are added or subtracted, respectively; this yields the transfer function of the comb-filter:

$$\underline{H}_{FIR} = 1 - \exp(-j\omega T_x); \qquad \left|\underline{H}_{FIR}\right| = 2 \cdot \left|\sin(\omega T_x / 2)\right|$$ FIR-filter

The designation **FIR-filter** (Finite Impulse Response) is due the impulse response being of finite duration. The magnitude of the frequency response is the magnitude of a sine-function with zeroes at 0 Hz and integer multiples of the reciprocal of the delay time T_x. This calculation is formally correct but inconvenient for illustrations, as **Fig. 2.28** shows. Similar problems are known from time-discrete signals if the sampling theorem is not adhered to: too low a sampling rate results in (usually undesirable) reverse convolution. In the present special case, however, the ambiguity due to the sampling is helping. Via the identity

$$\left|\sin(m\pi - \varphi)\right| \equiv \left|\sin(\varphi)\right| \text{ only for } m = \text{integer}$$

and a few intermediate steps, the FIR-transfer function may be converted into:

$$\left|\underline{H}_{FIR}\right| = \left|\sin\left(\pi \cdot \frac{f}{f_G} \cdot \frac{d}{M}\right)\right|$$ FIR-filter, reformulated

Herein, d represents the distance between the plucking point and the bridge, and M is the length of the open string (scale). For the fretted string, the scale needs to be applied here, as well, because it is included in the formula for the propagation speed of the wave. If the open string is plucked precisely in the middle, the long-term spectrum holds only odd harmonics – the zeroes of the sine-function are located at the even harmonics. The closer the plucking point is to the bridge, the wider the minima of the envelope are spaced. The conversion only holds in the steady-state part (discrete line-spectrum) but not for the transitory process. This is a basic condition for every transfer function, though: it always holds for the steady state only. Furthermore, we need to consider that the delays in the above model are frequency-independent – dispersion is not (yet) emulated. Spread-out spectra require, instead of simple delay lines, **all-passes** that approximate the string dispersion in the frequency response of their delay (Chapter 2.8.4).

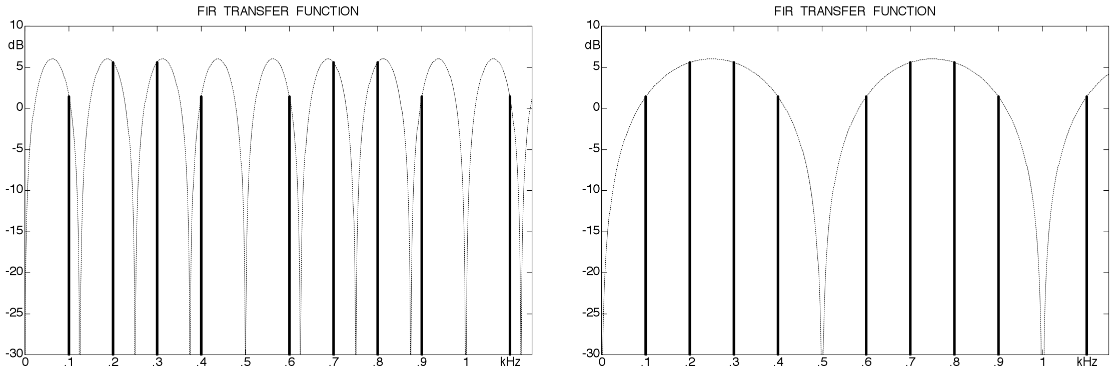

Fig. 2.28: FIR-filter frequency response (magnitude, ---) and filtered line spectrum for $d = M/5$. The lines shown are identical in both graphs; the graph on the right shows the transformed FIR-transfer-function.

In **Fig. 2.29** we see the measurements for a plucked E_2-string. The distance between plucking location and bridge amounted to d = 4,7 cm and 1,5 cm, respectively. From the results, the first minimum of the comb-filter calculates as 1,1 kHz and 3,5 kHz, respectively. In the low frequency region, the comb-filter structure is clearly visible in the spectral envelope – it is however perturbed by strain-wave resonances (Chapter 1.4, marked via dots). In anticipation of Chapter 2.8.4, Fig. 2.29 already includes the dispersive spreading of the spectral envelope. In addition, further selective damping mechanisms have an effect, especially in the high frequency domain. The associated causes will be elaborated on in Chapter 7.

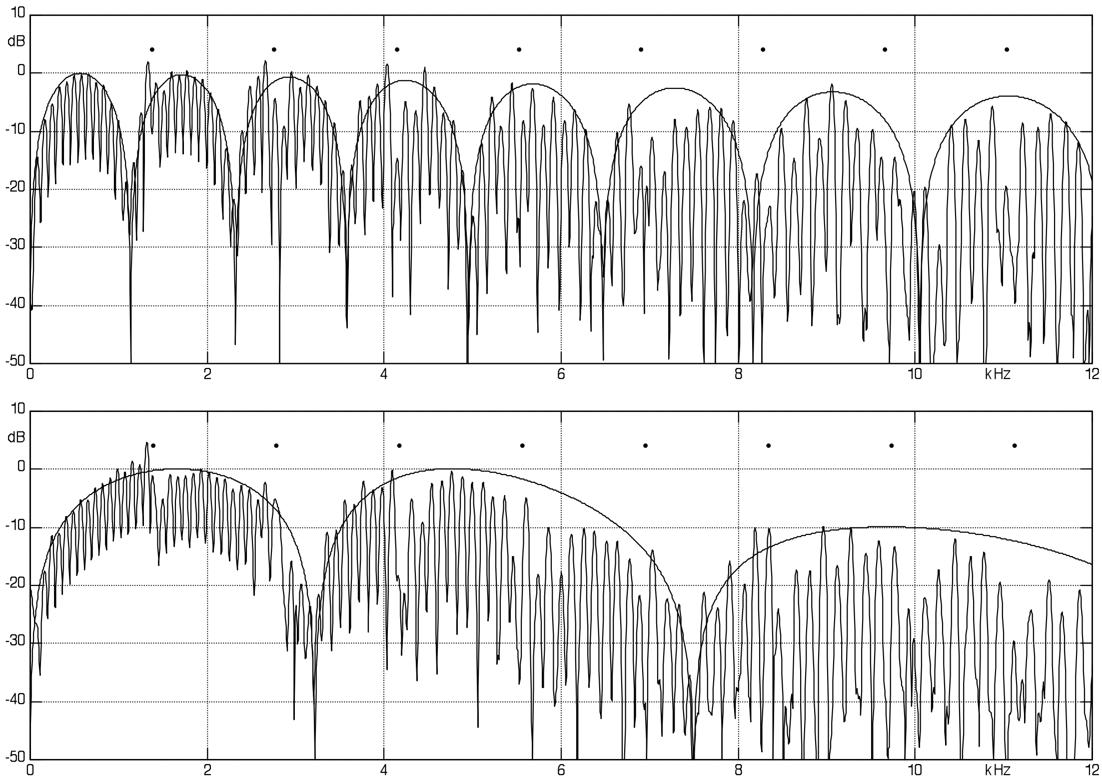

Fig. 2.29: Measured spectra; E_2-string (impulse excitation), d = 4,7 cm (top) and d = 1,5 cm (bottom). The shown envelope was spread out (dispersion) and slightly attenuated towards the high frequencies. The two measurements were taken with two different E_2-strings (OVATION Viper EA-68).

 Translation by Tilmann Zwicker

While the FIR-filter determines the spectral envelope, the recursive filter defines the frequency of the individual spectral lines. The impulse response of a recursive filter is of infinite length, which is why the term **IIR-filter** (Infinite Impulse Response) is common for this filter type. With both reflection coefficients being equal to 1, a short excitation impulse would circulate in the loop indefinitely without attenuation; such a filter is called borderline stable. Real strings have reflection coefficients of <1; the impulse-shaped excitation therefore decays over time. For a run through the full loop, both reflection coefficients act in multiplicative manner ($R{\cdot}r$).

Given $R{\cdot}r$ = 0,9, for example, the height of the impulse decreases e.g. from 1 to 0,9 for a single loop, to 0,81 for a double look, and to $0{,}9^n$ for an n-fold loop. The amplitudes of the impulses following each other with a distance in time of T represent a geometric progression; for $R{\cdot}r < 1$, the term used is **exponential decay**. Chapter 1.6 had already included quantitative statements regarding the decay process; for the guitar string, the loop coefficients are very close to 1 (e.g. 0,993). In the FIR-filter only a single reflection occurs, and therefore $r = -1$ may be used with very good approximation. However, in the IIR-filter, the loop is run through an infinite number of times, and consequently this approximation is not allowable.

Chapter 2.5 had shown that the reflection coefficient is not constant but frequency-dependent. The reason are resonances in the bridge and the nut (or fret) formed from a combination of springs and masses. These springs and masses are not necessarily all found within the bridge (or the nut or fret) but may be located e.g. in the neck of the guitar and act on the nut [8]. When integrating a frequency-dependent reflection coefficient into the SFD (Fig. 2.17), we need to pay attention to the fact that the system shown as circle ($R{\cdot}r$) becomes a filter that way: $R \cdot r(j\omega)$ is the frequency dependent transfer function of this **reflection filter**. The decay time-constant for each partial results from the loop-delay-time T (frequency dependent if the dispersion is considered), and from $R \cdot r(j\omega)$.The SFD (Fig. 2.27) does not consider the reason for the damping: it is the *overall* damping that is modeled via $R \cdot r(j\omega)$. If required, several individual filters may be connected in series, for example to be able to model the internal string damping in a separate subsystem.

Ahead of the input designated with E in Fig. 2.27 we need to position the **plectrum filter** that shapes the real excitation force from the ideal step (or from an impulse). The **piezo-filter**, or – for acoustic guitars – the body- and radiation-filter follows the output A. The structure-borne sound path is not modeled herein. If we think of the nut merely as a vibration absorber, this is not necessary, either: the damping caused by the nut is considered in $R{\cdot}r$, after all. However, part of the vibration energy flowing into the nut might be radiated, or fed back to the string via the bridge – something that necessarily would have to work in reverse, as well. The dilatational waves discussed in Chapter 1.4 use a similar bypass (albeit directly via the string).

Additional recursive loops enable a simple emulation of such parallel paths. It should be emphasized again, though, that this does not automatically make for a correct representation of the energy flows. In the SFD, a summation point adds two signals (e.g. two forces), but it does not model the impedances – these would a have to be considered separately depending on the circumstances.

