The string mentioned in the example having a fundamental frequency of 100 Hz, and the string shortened by half (fundamental frequency 200 Hz) each generate a tone designated T100 and T200, respectively. Played one after the other in direct comparison, T100 and T200 sound very similar – this is not actually surprising since the frequencies of the partials contained in T200 represent a subset of those contained in T100. This example may be extended by subjecting the halved string (T200) to another halving (T400). The resulting frequencies of the partials (400 Hz, 800 Hz, 1200 Hz, etc.) are again a subset of the frequencies of the partials contained in T100 and T200. Further halving of the string length gives corresponding results. All notes generated by such halving (or doubling) sound very similar, although their pitches differ markedly. Since the frequency relation generated by halving and doubling of the string lengths (2:1 and 1:2, respectively) are designated **octaves** in the musical context, the resulting notes are called **octave-related**. The high degree of auditory relationship between two notes distanced by an octave has led to designating such notes with the same letter. For example, the **reference note** used for tuning to standard ("concert") pitch is internationally as a rule designated A_4, with the note one octave above being designated A_5. However, depending on the national context there are also variations to this system of designations, e.g. a^1 (or a'), and a^2 (or a''), respectively.

8.1.1 The Pythagorean tonal system

Continued halving of the string-length is a first step towards generating related notes of differing fundamental frequency. Following this approach, we find notes with corresponding frequencies of partials also when **reducing the string-length to one third**. The partials of the resulting note (designated T300) are located at 300 Hz, 600 Hz, 900 Hz, 1200Hz, etc. However, compared to T200 now only the frequencies of every other (even-numbered) partial is in correspondence, namely 600 Hz, 1200 Hz, etc. (**Fig. 8.2**). The fundamental frequency of the string reduced to 1/3rd in length relates to the fundamental frequency of the halved string, as would 3:2; this frequency relation (frequency interval) is called, in musical terms, a **fifth**. For the associated notes, the concept of **fifth-relationship** is derived from this. Compared to the octave-relationship, the fifth-relationship is less pronounced.

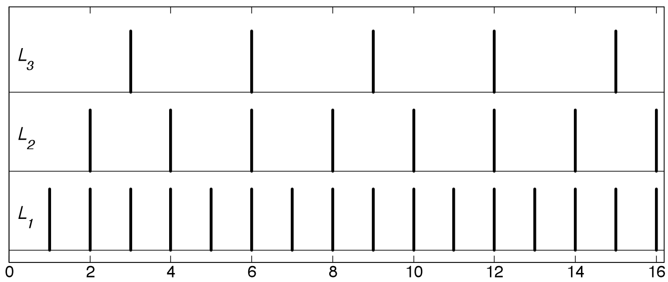

Fig. 8.2: Spectra of partials of strings with the relative lengths: L_1 = 1, L_2 = 1/2, L_3 = 1/3. Abscissa: normalized frequency; ordinate: amplitudes (arbitrary)

Applying jumps of fifths and octaves in combination allows for the generation of a multitude of notes that all are more or less related. Already in the ancient world a tonal system (among many others) was constructed from octave- and fifth- intervals; after its protagonist Pythagoras (ca. 530 B.C.), it is named the **Pythagorean tonal system**. In theory, an infinite number of different notes could be generated with it. However, in practice we arrive at a prominent end point after 12 jumps of one fifth each: after 12 subsequent intervals of one fifth each, the resulting frequency relationship is $1{,}5^{12} = 129{,}74_6$. This brings it close to the 7th octave, the frequency relationship of which amounts to $2^7 = 128$.

The small difference between these two values of 129,74$_6$ / 128 = 1,013$_6$ is called the **Pythagorean comma** in music theory. From the sequencing of fifths, and from octave shifts, all notes of Western music can be generated. In this approach, the frequencies of the notes positioned at a distance of a fifth are shifted by a number of octaves until all frequencies are located within one base octave. Starting from the arbitrarily chosen initial frequency 100 %, the following rounded (!) frequencies result (in order to be able to more easily interpret the frequencies, they are given in % to begin with; the corresponding frequencies are listed in Chapter 8.1.3):

100 –	150 –	225 –	338 –	506 –	759 –	1139 –	1709 –	2563 –	3844 –	5767 –	8650 –	12975	%.
100	150	113	169	127	190	142	107	160	120	180	135	203	%.
C	G	D	A	E	H	F#	C#	G#	D#	A#	E#	B#	

The first line in this table holds the ascending frequencies of fifths, the second line includes the corresponding frequencies in the base octave. The designation of the notes is given in the third line (# stands for 'sharp'). For example, 2563 % needs to be shifted (towards lower frequencies) by four octaves in order to arrive at 160%: 2563 / 2^4 = 160. Rearranging the frequencies in the second line in monotonously ascending order, the sequence of frequencies of a scale results (values rounded off):

100 –	107 –	113 –	120 –	127 –	135 –	142 –	150 –	160 –	169 –	180 –	190 –	203	frequency / %
C	C#	D	D#	E	E#	F#	G	G#	A	A#	B	B#	note-designation

Besides the ascending sequence of fifths, the descending sequence of fifths may also be generated: again neighboring notes are fifth-related. In correspondence to the example above, the initial frequency 100% would have to be repeatedly divided by 3/2: 67 %, 44 %, etc. With suitable octave shifts (towards higher frequencies), again a scale results – with calculated frequencies that slightly differ from the ones given above, though.

In the classical **Pythagorean tonal system**, not all of the notes calculated above were employed. Starting from the keynote C, users made do with 5 ascending fifths (C-G-D-A-E-B) and one descending fifth (F). They were able to form a **scale** that way:

1	$Q^2/2$	$Q^4/4$	$Q^{-1}\cdot 2$	Q	$Q^3/2$	$Q^5/4$	2
C	**D**	**E**	**F**	**G**	**A**	**B**	**C'**
1\1	8\9	64\81	3\4	2\3	16\27	128\243	1\2

In this table, Q represents the interval of the fifth♣ (frequency ratio 2\3); the corresponding exponent indicates the number of the required jumps of a fifth each. From the denominator, we can take the number of the additionally required octave shifts. $Q^5/4$ indicates 5 fifth-jumps towards higher frequencies, and subsequently 2 octave-shifts (2^2 = 4) towards lower frequencies. The third line yields, referenced to the keynote, the frequency relation as a fraction. The notes of the scale given above, and their frequency relation (interval), is designated according to their place number:
C = prime, D = second, E = third, F = fourth, G = fifth, A = sixth, H = seventh, C' = octave.

♣ To specify the frequency relations in an **interval-designation**, two different styles are customary: for the fifth e.g. 2:3 but also 3:2. Both relations are self-explanatory, while the letter-designation (C-G) does not unambiguously identify which one of the two is the lower note. In the following, the lower note is always positioned first (to the left) as is usual for axis-scaling. However, following through with this train of thought would result in fractions that are smaller than 1, such as e.g. $f_{C1} : f_{G1}$ = 2:3 = 0,666... While this representation is in itself correct, it is in contradiction with the practice of indicating intervals with number that are larger than 1. This contradiction is resolved in the following via using the back-slash (as used in Matlab): $f_{C1} \backslash f_{G1}$ = 2\3 = 1,5.

The terms are related to numeration in Latin: primus, sekundus, tertius, quartus, etc. In its precise meaning according to the theory of harmony, these expressions designate the *distances* between two notes (inter-vallum = space between palisade beams), but in everyday use they also represent the names of notes: *the fourth on the C-scale is an F*. Distance in the above sense means to indicate the distance to the root note i.e. the ratio of the frequency of the note in question (e.g. an F) to the frequency of the keynote; in this example it is 3\4, corresponding to a fourth. It is also possible to form the ratio of two notes directly neighboring on the scale; this yields:

$f_C \backslash f_D = 8\backslash 9$; $f_D \backslash f_E = 8\backslash 9$; $f_E \backslash f_F = \mathrm{HT}$; $f_F \backslash f_G = 8\backslash 9$; $f_G \backslash f_A = 8\backslash 9$; $f_A \backslash f_B = 8\backslash 9$; $f_B \backslash f_{C'} = \mathrm{HT}$;

Of these 7 frequency ratios, 5 correspond to a so-called **whole-step** (‘whole note’, ‘whole tone’), specifically C-D, D-E, F-G, G-A, A-H. The remaining two intervals of neighboring notes are **half-steps** (HT, ‘half notes’, ‘semi-tones’, ‘half-tones’). In Pythagorean tuning, the frequency ratio in a whole step amounts to 8\9 = 1,125, and the one in a half step (E-F, B-C) **HT** = 243\256 = 1,0535. The resulting scale is called **diatonic scale** because it is comprised of two different steps (namely whole-step and half-step). As supplemental information, ‘Pythagorean tuning’ should be indicated – there are many different tunings, after all.

N.B.: with respect to the **note that is internationally designated B**, there is a particular idiosyncrasy when the German language is used: there, this note is designated **H**. Originally (in fact: obviously), letters (starting with A) formed the names of the notes in the scale: A-B-C-D-E-F-G. However, medieval hexachord theory required (on top of the B as mentioned above) a second note half a step lower. In order to distinguish between the two, the designations *B-quadratum (B-durum)* and *B-rotundum (B-molle)* were introduced – derived from the angular (hard, *durum*) and round (soft, *molle*) writing styles of the letter *b*. The angular b mutated to an h … and now musicians in Germany, Austria, and the German speaking part of Switzerland found themselves with a peculiarity that continues to lead to (sometimes serious) complications when communicating internationally.

The diatonic scale as introduced above consists of 5 whole-steps and 2 half-steps. Each one of the whole-steps can pythagoreically be subdivided into two half-steps – however this may be done in two different ways. In the international note designations, half a step upwards is indicated with adding the syllable “sharp” to the note, and half a step downwards by adding the syllable “flat”. The diminished D is called **D-flat** (D*b*, with the *b* standing for ‘diminished’), the augmented C is **C-sharp** (C#). It has already been shown that all notes can be generated by using upwards-fifths and downwards-octaves in the Pythagorean sense:
C–G–D–A–E–B–F#–C#–G#–D#–A#–E#–B#.
However, all notes may just as well be generated via downward-fifths and upward-octaves:
C–F–B*b*–Eb–A*b*–D*b*–G*b*–C*b*–Fb–B*bb*–E*bb*–A*bb*–D*bb*.
The notes B*bb*, E*bb*, A*bb* and D*bb* result from diminishing B, E, A, D by *two* half-steps, respectively.

Fig. 8.3 shows the keynote frequencies of these two Pythagorean-chromatic scales. Due to the Pythagorean comma, no frequencies in a pair in the sequence of upwards-fifths and downwards-fifths are the same (except for the starting pair). If we limit ourselves to diminishing by a *single* half-step, a scale of 21 steps results: each of the 7 diatonic steps C-D-E-F-G-A-B is allocated a lower and a higher half-step. This 21-note tonal system was actually the basis for keyboard instruments – however it was deemed too complex.

Many musicians therefore simplified the scale by enharmonically equating similar notes. The resulting **12-step Pythagorean-chromatic scale** is indicated on the top of Fig. 8.3 via squares. Only a single half-step is introduced each between all whole-steps, but the half-tone distances are of different size, as is clearly visible (□—□).

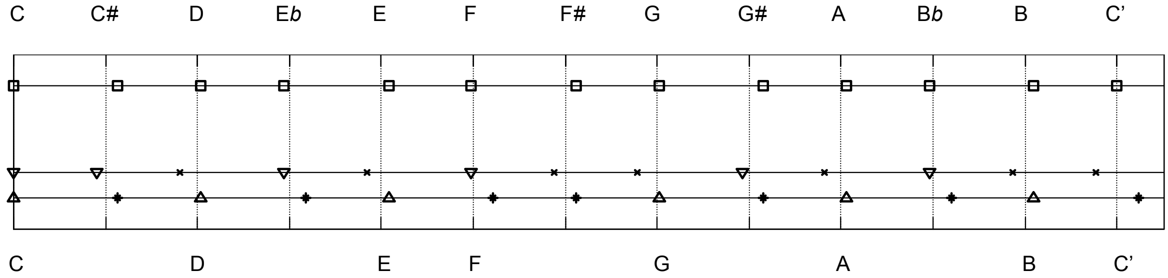

Fig. 8.3: Fundamental frequencies of the Pythagorean-chromatic scale, shown on a logarithmic frequency axis.
Δ = deduced from the first 6 upwards-fifths jumps; ∇ = deduced from the first 6 downwards-fifths jumps;
×,∗ = remaining 7 fifths jumps; □ = used in medieval times as chromatic scale.
The scale with equal temperament developed around 1700 is indicated with dashed vertical lines (8.1.3).

The different half-step distances complicate changing keys: the second (C-D) based on the keynote C has a larger frequency difference that the one based on C# (C#-E*b*), and other intervals (e.g. C-E, G#-C) meet a similar fate. Depending on the specific case, the flawed consonance when two notes are played simultaneously may be another problem. The fundamental thought behind the Pythagorean tuning was the note-relationship based on fifths and derived from the sequence of partials. Well meant that is – but you know how things are with relatives: as the distance grows, the similarities wane. **Fig. 8.4** schematically shows the frequencies of the partials for the prime (C) and the third (E). If, in simultaneous playing of the two notes, individual partials get to lie (frequency-wise) in immediate vicinity, **beats** may become audible. An example would be the 5th partial of the prime (C) and the 4th partial of the Pythagorean third (E_p).

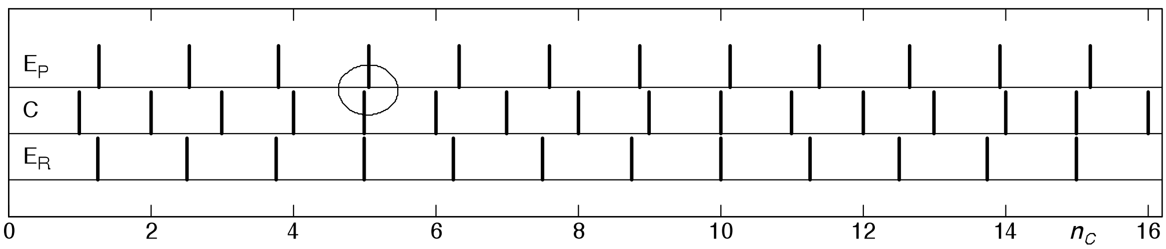

Fig. 8.4: Spectrum of partials of the notes C (prime) and E (third). Beats are generated between the 5th partial of the prime and the 4th partial of the Pythagorean third (E_P), due to the small frequency difference. For the pure third (E_R) the corresponding frequencies of the partial are identical. Abscissa: normalized frequency of the partials of the prime.

Beating happens when two mono-frequent notes of equal amplitude and similar frequency are played at the same time (i.e. they are added). Every note from a guitar consists of a multitude of (mono-frequent) partials, each of which is, individually considered, sine-shaped (a cosine-oscillation has the *shape* of a sine, as well). The 5th partial (= 4th overtone) of an ideal string vibrating at 100 Hz has the frequency of 100 Hz x 5 = 500 Hz, the 4th partial of the third according to Pythagorean tuning is at 126 Hz x 4 = 504 Hz. The frequency difference of the two partials is 4 Hz.

If we now regard merely the oscillation of the sum of the two partials, a figure similar to **Fig. 8.5** (center) results. The phase difference of the two partials fluctuates with the rhythm of the difference frequency, and amplification and cancellation alternate with the same rhythm. Given sufficient levels, *one single* partial with rhythmically fluctuating (i.e. beating) loudness is heard rather than two partials of almost equal pitch.

Interpreting the summation-curve (middle section of Fig. 8.5) is facilitated by reformulation towards a multiplicative operation:

$$\cos(2\pi f_1 t) + \cos(2\pi f_2 t) = 2 \cdot \cos(2\pi f_\Delta t) \cdot \cos(2\pi f_\Sigma t); \quad f_\Delta = \frac{f_2 - f_1}{2}; \quad f_\Sigma = \frac{f_2 + f_1}{2}$$

In this product-representation, f_Σ stands for the frequency of a cosine-oscillation with its amplitude changing "with the rhythm of the difference frequency f_Δ". The above example has f_Σ = 502 Hz, thus it lies exactly in between the primary frequencies f_1 und f_2. The term "difference frequency" should be used with care: it is calculated as f_Δ = 2 Hz, this is *half* the frequency distance between f_1 und f_2. However, the maximum of the beat-envelope appears (amount!) with double this frequency i.e. twice per f_Δ-period. The above beating with 500 Hz and 504 Hz as primary frequencies may therefore be seen as a tone at 502 Hz featuring 4 envelope maxima and 4 envelope minima per second. It therefore becomes louder and softer 4 times per second. The auditory effect of a beating of partials is difficult to predict – it may even be inaudible (despite its physical presence) due to masking by neighboring frequency components. If it indeed is audible, it may sound pleasant or displeasing. During many centuries the opinion was held that any beating of partials is undesirable, resulting in the beat-free **just intonation** (Chapter 8.1.2).

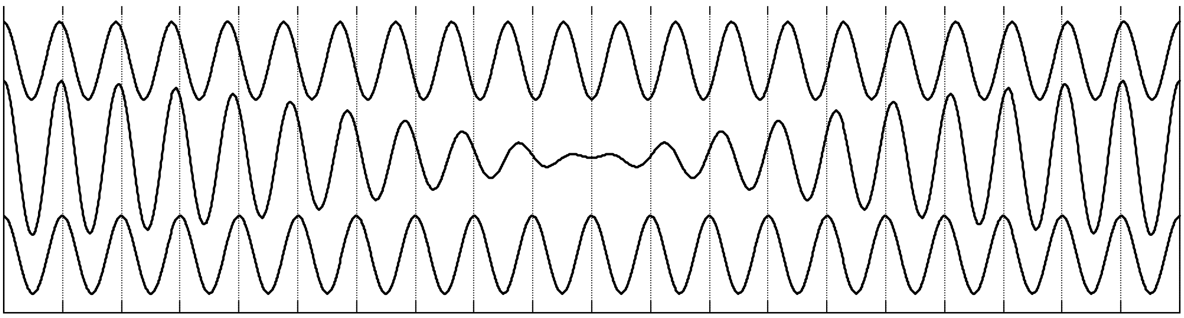

Fig. 8.5: Two cosine oscillations (top, bottom) slightly different (5%) in frequency, and their sum (middle). The curves are of equal phase at the left and right boundaries of the figure, and in opposite phase in the middle. Same-phase addition results in doubling of the amplitude (constructive interference), opposite-phase addition leads to cancellation (destructive interference). Abscissa: time.

8.1.2 Just intonation

In this context, *harmonic* and *natural* also stand as synonyms for *just* – the rationale being that nature herself allegedly had shown the way in the form of integer frequency ratios of the partials. The term *divine tuning* therefore is not far off, creating work for philosophers and esoterics, but mainly for mathematicians ... who not necessarily were musicians.

