If we now regard merely the oscillation of the sum of the two partials, a figure similar to **Fig. 8.5** (center) results. The phase difference of the two partials fluctuates with the rhythm of the difference frequency, and amplification and cancellation alternate with the same rhythm. Given sufficient levels, *one single* partial with rhythmically fluctuating (i.e. beating) loudness is heard rather than two partials of almost equal pitch.

Interpreting the summation-curve (middle section of Fig. 8.5) is facilitated by reformulation towards a multiplicative operation:

$$\cos(2\pi f_1 t) + \cos(2\pi f_2 t) = 2 \cdot \cos(2\pi f_\Delta t) \cdot \cos(2\pi f_\Sigma t); \quad f_\Delta = \frac{f_2 - f_1}{2}; \quad f_\Sigma = \frac{f_2 + f_1}{2}$$

In this product-representation, f_Σ stands for the frequency of a cosine-oscillation with its amplitude changing "with the rhythm of the difference frequency f_Δ". The above example has f_Σ = 502 Hz, thus it lies exactly in between the primary frequencies f_1 und f_2. The term "difference frequency" should be used with care: it is calculated as f_Δ = 2 Hz, this is *half* the frequency distance between f_1 und f_2. However, the maximum of the beat-envelope appears (amount!) with double this frequency i.e. twice per f_Δ-period. The above beating with 500 Hz and 504 Hz as primary frequencies may therefore be seen as a tone at 502 Hz featuring 4 envelope maxima and 4 envelope minima per second. It therefore becomes louder and softer 4 times per second. The auditory effect of a beating of partials is difficult to predict – it may even be inaudible (despite its physical presence) due to masking by neighboring frequency components. If it indeed is audible, it may sound pleasant or displeasing. During many centuries the opinion was held that any beating of partials is undesirable, resulting in the beat-free **just intonation** (Chapter 8.1.2).

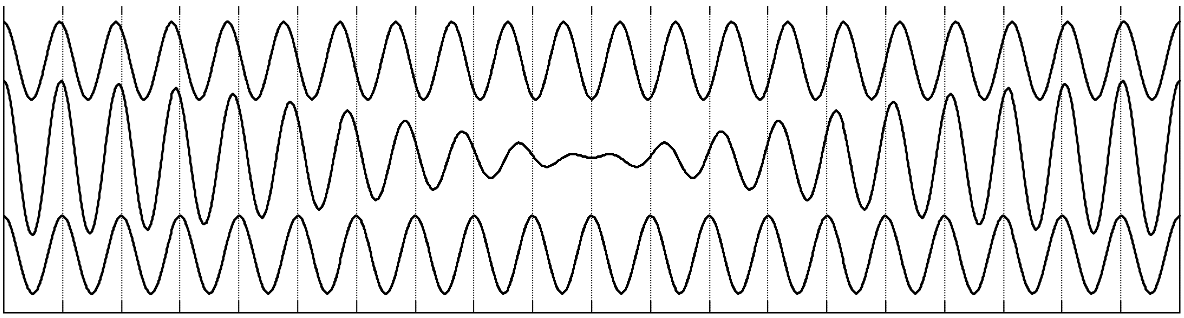

Fig. 8.5: Two cosine oscillations (top, bottom) slightly different (5%) in frequency, and their sum (middle). The curves are of equal phase at the left and right boundaries of the figure, and in opposite phase in the middle. Same-phase addition results in doubling of the amplitude (constructive interference), opposite-phase addition leads to cancellation (destructive interference). Abscissa: time.

8.1.2 Just intonation

In this context, *harmonic* and *natural* also stand as synonyms for *just* – the rationale being that nature herself allegedly had shown the way in the form of integer frequency ratios of the partials. The term *divine tuning* therefore is not far off, creating work for philosophers and esoterics, but mainly for mathematicians ... who not necessarily were musicians.

Just intonation – the term rings of *teachings of justice & purity*, and expressions such as *fairness*, *correctness*, or *well justified* come to mind – the opposite of *unjust*, *wrong*, or *unjustified*, and thus anything not conforming to the *just intonation* could in any case only be heresy. It is easily imaginable how hordes of mathematicians have deduced justifications for this or for that intonation … generating tables with an accuracy up to 12 figures! Or, rather: tables with 12 decimals, since the actual accuracy may have been a bit of an issue [Barbour]. Irrespective of any (not infrequently occurring) calculation- and rounding-errors: given a 1-m-long monochord string, 12-decimal-accuracy implies a length-tolerance of no more than 0,001 nm. Just to compare: the wavelength of visible light amounts to around 600 nm. Specifying pitch deviations with an “accuracy” of 1/10000000th of a cent is similarly nonsensical.

The **just intonation** may be traced back to ancient times. Two doctrines of thought emerged from the Pythagorean school (that originated around 530 BC): the **canons regular** (canon = rule, law) advocated the conservative opinion, while the **harmonists** gave priority to euphony, even if that required modification of mathematical laws of nature. The canonical doctrine regarded the frequency ratio 6 : 8 : 9 : 12 as “holy matrimony” between the forth and the fifth (**Fig. 8.6**) with the major second (full step F-G) being the result. Simbriger/Zehelein give an astounding assessment for this approach: *we have already met this grouping of notes in primitive music; with the Pythagoreans, we find that same basic occurrence substantiated and sanctioned with the background of advanced civilization.* There you have it: if – as a musician or listener – you recognize certain intervals as harmonic/consonant, then that’s primitive ado. However, if you smudge some divine-cosmic-mystical mumbo-jumbo around that finding, it takes its place in high culture.

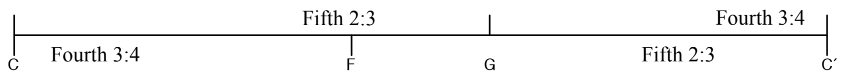

Fig. 8.6: The "holy matrimony"

Still: despite some massive mystical sanctioning it was not possible to hide that the use of Pythagorean intonation made some chords sound less than pleasant. Young J.-apprentice: "oh honorable master Y.: them chords, they will not sound – try as I might! Those fifths and thirds, they fail to soothe us.” Y.: “Do or do not: there is not try … but quiet now be, young one; in a special realm here taken we are. Let be it, for divine this is – of The Force” ☺. Many will have conformed to this sage advice from a long time ago and a galaxy far, far away … but some went public. In the olden days, on this planet, that could well lead to premature termination under artificially elevated ambient temperature – or it could open the door to eternal fame and glory. Or both. **Didymos** (Didymus) and **Ptolemy**, Alexandrinian savants by trade (and, to begin with, both by all means proponents of the Pythagorean third), evidently found the silver bullet (at the time probably the silver arrow). They replaced the Pythagorean third (based on the divine fifth) by an at-least-as-divine relation of whole steps: the major third – in Pythagorean intonation the frequency interval 64\81 = 1,2656 – was shifted to 4\5 = 1,2500 in the so-called Alexandrinian system. Didymos borrowed the minor third (27\32 = 1,1852) from the Pythagorean system, and Ptolemy modified it to 5\6 = 1,2000. In principle, anyway. Looking closer, we find [e.g. in Barbour] two didymian intonations, and no less than 7 ptolemyan intonations. Nevertheless, the foundation block for the just intonation was laid.

Studying literature, it is easy to come to the impression that (as mentioned above) something divine is connected to the just intonation. However, as confusion grows, the realization does manifest itself that it must in fact be a kind of polytheism. Barbour defines *just intonation* as: based on octave (1\2), fifth (2\3) and major third (4\5); the intervals themselves are designated *just* (or *pure*), as well.

Elsewhere, however, Barbour extends the term *just intonation* to: based on octave (1\2), fifth (2\3), fourth (3\4), major third (4\5), and minor third (5\6). Other authors even designate as *pure intervals* all intervals the frequency ratios of which correspond to the whole-numbered ratios of the frequencies of the first 16 partials. All intervals? Well, almost … those ratios that fit to some degree, anyway. But not the 7th , 11th , 13th , and 14th partials! Of course not. Valentin substantiates: *the miraculous, natural, and therefore not worked-out order of the whole system stems from the sequence of the composition of just intervals contained in these notes that – with a suitable octave transposition – yield our whole scale system.* The 7th, 11th, 13th, and 14th partials are the "black sheep"; nature finds space for something like that, too. Only for C-F# (or C-G*b*) no fitting frequency ratio at all could be found in the natural order. Therefore, the devil had to be called in as the usual suspect – only he/she could have smuggled in such an inconvenient, devilish interval (Tritonus, Diabolus in Musica). The question: "how could God allow this …" again created many workplaces for philosophers (compare Theodizee), but this would go beyond the scope of scientific considerations.

The just intonation derives its rationale from the whole-numbered frequency ratios of the first 16 partials. But why exactly 16 partials? That's because the 16th partial is exactly 4 octaves above the fundamental. But why then not just 1 or 2 or 3 octaves? That would be because that way you could not yet generate a chromatic scale. Moreover, wind instruments can just about reproduce the 16 "natural tones" (Eigen-tones, partials). The peculiarity of the tritone with its 45\64-ratio was justified on the basis of this fact that about 16 but not those 64 Eigen-tones could be generated. **Fig. 8.7** shows the frequency ratios of a just-intoned scale. Besides the devil's interval, there indeed is nothing fishy in there: numerators and denominators are integers between 1 and 16. The **major third** C-E that would with the Pythagorean intonation carry beats – it now is beat-free (compare to Fig. 8.4).

1 15\16 8\9 5\6 4\5 3\4 45\64 2\3 5\8 3\5 5\9 8\15 1\2
C – D*b* – D – E*b* – E – F – G*b* – G – A*b* – A – B*b* – B – C'

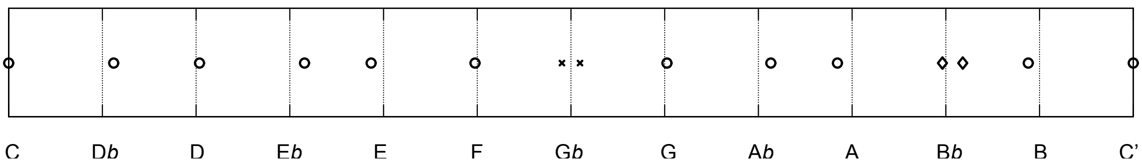

Fig. 8.7: Just intonation (Mersenne's lute tuning Nr. 2). The tritone was given also as F# with 32\45, for the Bb also 9\16 are found instead of 5\9.

Besides C-E, the combinations F-A and G-B (with 4\5) also make for a beat-free major-third interval. For the **minor thirds**, however, differences already appear now: E-G, A-C, and B-D yield 5\6, but D-F yields 27\32. Looking at the **fifth**-intervals: C-G, E-B, F-C, G-D and A-E yield 2\3, but D-A → 27\40. The **whole-step** intervals are at 8\9 or 9\10; the **half-step** intervals on the C-major scale are at 15\16, with the remaining (chromatic) half-step intervals at 24\25, 25\27 or 128\135. Despite the legitimization by nature herself, this gave opportunities for mockers: are you still learning, or do you play with a special intonation system?

It wasn't that these dissonances remained hidden to the working musicians. The latter knew about them, limited their music-making to a few keys, and tried to give a wide berth to the *howling-wolf intervals.* Alternatively, instruments could be built that divided every octave into 21 in-between notes. And if that didn't suffice: J. M. Barbour lists a plethora of other divisions, for example: the 31-division (Fibonacci-sequence), the 53-division (Bosanquet-harmonium), and don't you forget that *the 118-division has both fifths and thirds that are superlative (0,5 cent flat and 0,2 cent sharp, respectively)*.

There we are! … and there we go. On top of all that, other just (!) intonations were developed for the twelve-section octave, as well – which makes Barbour infer: **<u>the</u> just intonation does in fact not exist; rather, there are many different just intonations, with the best being the one that comes closest to the Pythagorean intonation**.

As desirable as "just" (or "pure") intervals may be in polychoral play: for intervals succeeding each other errors do cumulate. Take, for example (and see **Fig. 8.8**), Jimi Hendrix' "Hey Joe" (to solidly arrive back again in more modern times): the accompaniment first climbs down a minor third from E to C (or that could be interpreted as climbing up a minor sixth) and then runs through 4 jumps of fourths: E → C – G – D – A – E. Given just intervals, one revolution gets us this:

$$\frac{4\cdot 3\cdot 3\cdot 3\cdot 3}{5\cdot(2\cdot 2)\cdot 2\cdot(2\cdot 2)\cdot 2} = \frac{324}{320} = 1{,}0125 \mathrel{\hat{=}} +21{,}5\,\text{cent};$$ C ↓ E = 5\4, fifth = 2\3, octave jump = 2\1.

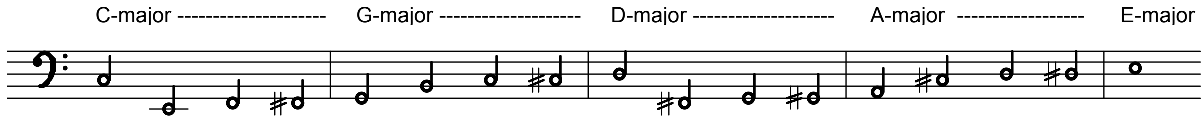

Fig. 8.8: Jimi Hendrix / Noel Redding: bass chromatic in "Hey Joe".

On the basis of just intervals, the full revolution of a cadence (lasting about 12 seconds in the original tempo) would lead to a frequency increase of 1,25% – after one minute, that would already make for no less that half a step. To execute every revolution at exactly equal pitch, e.g. the step from D to A (fifth) would have to be performed with the deviating ratio of 27\40. That, however, would mean a conflict with the pure (just) school of the first 16 natural notes.

Another "law of nature" (one that chalked up some success in architecture) is the **golden section** (or **ratio**). However, for Barbour the "golden tonal system of theoretical acoustics" is worth only a few lines. His bottom line: a jack-o'-lantern (ignis fatuus).

8.1.3 Tempered tunings

In music, the term temperament is used synonymously with the term tuning. **Tempered tuning** is no pleonasm, though. It is the technical term for tunings that, on a small scale and in a targeted manner, deviate from global tuning rules. Early versions of the tempered tuning may be traced back to Giovanni Maria Lanfranco (1533); starting from just intonation, he proposed to slightly down-tune the fifths, and to up-tune the thirds just as much as was tolerable (in terms of the perceived sound). During subsequent eras, there were countless attempts to define this advice more precisely. Starting from empirical results (the fifth should cause one beat per second), via graphical designs, nomograms, scary formula, and versatile tables, the path led to the equal-temperament tuning that dominates today: the octave is divided into 12 equidistant half-note-steps – and that's it. That this seemingly simple rule has not been in practice much longer – that is probably due to its demanding a readiness to compromise. It does require, after all, detuning those just and highly consonant intervals (such as the fifth). Not all musicians show a corresponding capacity for suffering: the cellist Pablo Casals speaks of the *brainwashing of the tempered tuning*, and the violinist Carl Flesch allegedly was unable to play together with a piano (in tempered tuning) subsequent to a rehearsal with a string quartet.