There we are! … and there we go. On top of all that, other just (!) intonations were developed for the twelve-section octave, as well – which makes Barbour infer: **<u>the</u> just intonation does in fact not exist; rather, there are many different just intonations, with the best being the one that comes closest to the Pythagorean intonation**.

As desirable as “just” (or “pure”) intervals may be in polychoral play: for intervals succeeding each other errors do cumulate. Take, for example (and see **Fig. 8.8**), Jimi Hendrix’ “Hey Joe” (to solidly arrive back again in more modern times): the accompaniment first climbs down a minor third from E to C (or that could be interpreted as climbing up a minor sixth) and then runs through 4 jumps of fourths: E → C – G – D – A – E. Given just intervals, one revolution gets us this:

$$\frac{4\cdot 3\cdot 3\cdot 3\cdot 3}{5\cdot(2\cdot 2)\cdot 2\cdot(2\cdot 2)\cdot 2} = \frac{324}{320} = 1{,}0125 \;\hat{=}\; +21{,}5\,\text{cent}\,;$$ C ↓ E = 5\4, fifth = 2\3, octave jump = 2\1.

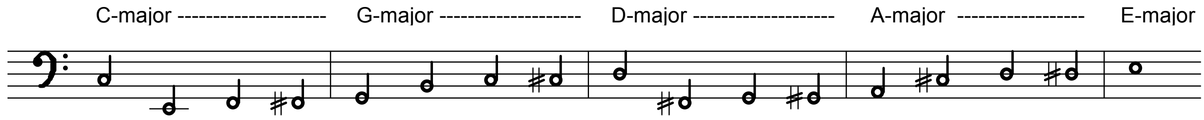

Fig. 8.8: Jimi Hendrix / Noel Redding: bass chromatic in "Hey Joe".

On the basis of just intervals, the full revolution of a cadence (lasting about 12 seconds in the original tempo) would lead to a frequency increase of 1,25% – after one minute, that would already make for no less that half a step. To execute every revolution at exactly equal pitch, e.g. the step from D to A (fifth) would have to be performed with the deviating ratio of 27\40. That, however, would mean a conflict with the pure (just) school of the first 16 natural notes.

Another “law of nature” (one that chalked up some success in architecture) is the **golden section** (or **ratio**). However, for Barbour the “golden tonal system of theoretical acoustics” is worth only a few lines. His bottom line: a jack-o’-lantern (ignis fatuus).

8.1.3 Tempered tunings

In music, the term temperament is used synonymously with the term tuning. **Tempered tuning** is no pleonasm, though. It is the technical term for tunings that, on a small scale and in a targeted manner, deviate from global tuning rules. Early versions of the tempered tuning may be traced back to Giovanni Maria Lanfranco (1533); starting from just intonation, he proposed to slightly down-tune the fifths, and to up-tune the thirds just as much as was tolerable (in terms of the perceived sound). During subsequent eras, there were countless attempts to define this advice more precisely. Starting from empirical results (the fifth should cause one beat per second), via graphical designs, nomograms, scary formula, and versatile tables, the path led to the equal-temperament tuning that dominates today: the octave is divided into 12 equidistant half-note-steps – and that’s it. That this seemingly simple rule has not been in practice much longer – that is probably due to its demanding a readiness to compromise. It does require, after all, detuning those just and highly consonant intervals (such as the fifth). Not all musicians show a corresponding capacity for suffering: the cellist Pablo Casals speaks of the *brainwashing of the tempered tuning*, and the violinist Carl Flesch allegedly was unable to play together with a piano (in tempered tuning) subsequent to a rehearsal with a string quartet.

Well, with direct access to the string and thus the pitch as a continuum, a violinist has the advantage of a completely free-wheeling intonation. The piano does not offer this possibility. If a growth of the number of keys into the infinite is to be avoided, the only remaining solutions are a highly key-specific temperament, or a universal equal-beating (equal tempered) tuning.

Intervals between notes are characterized by the corresponding frequency ratios. Using the equal temperament, 12 similar half steps succeed one another within an octave, with a geometric frequency sequence resulting: 1, *HS*, *HS·HS*, *HS·HS·HS*, etc. Here, *HS* indicates the half-step interval, the 12-fold repetition of which yields the just (pure) octave: $HS^{12} = 2$. With this, the frequency relation of directly neighboring notes (at half a step distance each) calculates as:

$$HS = \sqrt[12]{2} = 1{,}059463...$$ Half-step interval in the equal temperament

The 12th root of 2 … that is an irrational number. In the actual sense of the word it is a number opposing reason. That may also be why a queasy feeling crept up on many a music-theorist. 3\4 for the just-intonated fourth is specified by nature herself; the counterpart in equal temperament, on the other hand, defies – with HS^5 – all sanity. And yet the numerical differences are not all that big: 3\4 = 1,33333... , HT^5 = 1,33484... , that's a gap of no more than merely 0,1%. However when principles are at stake, the gods themselves fight in vain. And sure: the differences may indeed be larger for other intervals. The following table lists all notes and frequency ratios in the equal-temperament scale. Other notes are not defined i.e. there is no distinction between C#/D*b*, E#/F, A*bb*/G, B/C*b*, and so on.

C	C#=D*b*	D	D#=E*b*	E	F	F#=G*b*	G	G#=A*b*	A	B*b*	B	C'
0	1	2	3	4	5	6	7	8	9	10	11	12
1	1,0595	1,1225	1,1892	1,2599	1,3348	1,4142	1,4983	1,5874	1,6818	1,7818	1,8877	2

Table: Notes and frequency ratios in equal-tempered tuning. The second line yields the half-note steps, the third yields the frequency ratios rounded to 4 decimal places. Reference = C.

In German-speaking lands, the term *gleichschwebend* (= with equal beating) could be misinterpreted such that all intervals would cause similar beating. This is not the case. The English designation EQUAL TEMPERAMENT is not self-explanatory, either. It is the half-note steps that are equal (in terms of the frequency ratios), and not the beats. Also equal (in the sense of relatively equal) is the distribution of the Pythagorean comma into all 12 jumps of fifths. Occasionally, *well-tempered* is found as a synonym for *equal tempered*; this can probably be traced back to **J. S. Bach's** preludes and fugues that he published under the title "The Well-Tempered Clavier". However, presumably Bach's instruments were not intonated with equal temperament (equal beats), but according to Werckmeister. Andreas **Werckmeister** (*Musikalische Temperatur*, 1691) had developed a tuning that comes close to the equal-temperament tuning but is not identical. Already one century earlier (around 1596), Simon **Stevin** had built a monochord the half-step frequency ratio of which corresponded to the 12th root of 2 (i.e. 1,059…). Presumably this was the first such instrument in Europe [Barbour]. Almost at the same time (around 1636), Marin **Mersenne**♣ carried out comprehensive theoretical groundwork.

♣ 1492 Franchinus Gafurius: Theorica musicae
1533 Giovanni Lanfranco: Scintille di Musica
1596 Simon Stevin: Monochord mit HT = $2^{1/12}$
1691 Andreas Werckmeister, Musicalische Temperatur
1511 Arnolt Schlick: Book on organ-building
1544 Michael Stifel: Arithmetica integra, z.B. log
1636 Marin Mersenne: Harmonie universelle
1706 Johann Neidhardt: Gleichschweb. Temp.

In his chapter *Equal Temperament*, Barbour lists no less than overall 41 different tempered tunings: eventual success had many parents that presumably had to fight vehemently for recognition. Even today, bitter adversaries turn up who are bothered by beating, "unnatural" intervals, while proponents of equal temperament revel in unlimited modulations. **Guitar players** should better make sure they run with the latter group because their instrument is manufactured using equal-temperament tuning.

In order to unambiguously define the whole relational range, it is also necessary to specify an **absolute value** besides just the frequency *relations* of the notes on a scale. As the long-standing reference (concert pitch), a^1 – the so-called middle A (also designated a' or A_4) – is in service. Today, the standard tuning frequency is **440 Hz** while in past centuries there were significant deviations in the range between 337 Hz and 567 Hz. In Germany, the reference was fixed to 422 Hz in Berlin in 1752. The year 1858 saw a proposal for international standardization on the conference on concert pitch in Paris, followed – on the corresponding conference in Vienna in 1885 – by the adoption of 435 Hz. On the ISA-conference in London in 1939, this value was increased to 440 Hz, and confirmed in 1971 by an ISO-resolution (ISO = International Standard Organization). In conjunction with the standardization, it was suggested to use the reference pitch for interval signals in radio and television, and as dial tone for the telephone. This was not a successful marketing idea: for the telephone, check measurements in 2004 showed a 6% deviation. The following table gives some fundamental frequencies for notes tuned to equal temperament; reference for A_4 is 440 Hz.

C	C#=D*b*	D	D#=E*b*	E	F	F#=G*b*	G	G#=A*b*	A	B*b*	B
523,25	554,37	587,33	622,25	659,26	698,46	739,99	783,99	830,61	880	932,33	987,77
261,63	277,18	293,66	311,13	**329,63**	349,23	369,99	392,00	415,30	440	466,16	493,88
130,81	138,59	**146,83**	155,56	164,81	174,61	185,00	**196,00**	207,65	220	233,08	**246,94**
-	-	-	-	**82,41**	87,31	92,50	98,00	103,83	**110**	116,54	123,47

Table: Frequencies of tones tuned to the equal-temperament scale, referenced to A_4 = 440 Hz; rounded to two decimal places. The open strings on the guitar E_2, A_2, D_3, G_3, B_3, E_4 are in bold.

In order to obtain convenient specifications of small deviations from correct tuning, Alexander John Ellis defined (in 1885) the **cent** as the (supposed) pitch-atom:

$$1\,\text{cent} = 2^{1/1200} = 1{,}0005778 \qquad\qquad \text{Interval} = 3986\cdot\lg(f_2/f_1)\ \text{cent}$$

1 cent amounts to 1/100th of a half-step, or to the 1200th part of an octave. The frequencies 2000 Hz and 2001,155 Hz differ by 0,058% i.e. by 1 cent. Simbriger/Zehelein cite Preyer with the insight (questionable from a present-day perspective) that the hearing system was able to distinguish 1200 pitch steps between 500 Hz and 1000 Hz. Presumably, many a teacher scared away their pupils by demanding that the latter should be able to discern intonation errors of a 100th of a half-step. Chapter 8.2.2 has more on this topic.